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THE ORBIT OF A SATELLITE OF AN OBLATE PLANET

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Abstract. The paper gives a solution of the problem of motion of a particle in the potential field

$$V = -1/r + 2kP_2(\sin\theta)/r^3 + k'P_4(\sin\theta)/r^5,$$

where k and k' are small parameters. The first approximation is furnished by a V_0 that incorporates a major portion of the second spherical harmonic and leads to a closed solution with no secular variations of order k. The perturbations of this non-Keplerian intermediary orbit are derived here by the von-Zeipel modification of the method of Delaunay. The secular variations are carried up to orders k^2 and k', and the periodic variations to orders k and k'/k. A computational summary is included; the effect of the third spherical harmonic is appended.

1. *Introduction.* Until the advent of the artificial satellites, the restricted Particle-Spheroid problem has held the distinction of being the simplest of the unsolved problems of celestial mechanics. Under the assumption of axial and equatorial symmetry, the gravitational potential of an oblate planet may be approximated by

$$V = -1/r + 2kP_2(\sin\theta)/r^3 + k'P_4(\sin\theta)/r^5. \quad (1)$$

Here r is the radius vector, θ the declination, P_2 and P_4 the Legendre polynomials; the planetary constants k and k' are assumed to be small quantities of the first and the second orders respectively.

Since there is no exact solution of the problem of motion in spherical coordinates, attempts have been made to find a V_0 that leads to an exact solution and that approximates V better than the term $-1/r$. The author found such an intermediary orbit (Garfinkel 1958). In its simplicity it approaches the ellipse; in its accuracy it far surpasses the latter. A definitive solution of the problem must include the periodic variations of $O(k)$ and the secular variations up to $O(k^2)$. The author first solved the problem by the method of Poisson, based on the Taylor series expansion, in powers of k, of the equations of the variations of the constants. In this paper the results are re-derived by the more elegant and powerful method of Delaunay. The latter is based on a canonical transformation that removes the periodic terms from the hamiltonian. The solution includes as a special case the recent results of Brouwer, obtained by the application of the same method to the elliptic approximation $V_0 = -1/r$.

2. *The intermediary orbit.* It has been shown in the earlier paper that

$$V_0 = -1/r + 6k[c_1(\sin^2\theta - c_2)/2r^2 + c_3/r + c_4/r^3] \quad (2)$$

is the unique potential with the two properties: (1) It achieves the Hamilton-Jacobi separability in spherical coordinates and leads to a closed solution in terms of elliptic functions. (2) It preserves the gross features of V; i.e. the equatorial and the axial symmetry, the singularity at the origin, the vanishing at infinity, and the order relation $V + 1/r = O(k)$.

T. E. Sterne (1958) has treated the case $c_3 = 0$; the author has chosen the case $c_4 = 0$,

$$V_0 = -\mu/r + 3kc_1(\sin^2\theta - c_2)/r^2, \qquad \mu = 1 - 6kc_3, \quad (3)$$

which minimizes the number of elliptic functions in the solution. With this choice of V_0 as a first

approximation to V, the disposable parameters c_1, c_2, and c_3 have been determined so as to remove the secular variations of $O(k)$. The principal results of that paper will be briefly summarized here.

The equations of the orbit defined by (3) are:

$$
\begin{aligned}
E - e\sin E &= nt + \sigma,\\
r &= a(1 - e\cos E),\\
\tan\frac{v}{2} &= \left[\frac{1+e}{1-e}\right]^{\frac{1}{2}}\tan\frac{E}{2},\\
u &= \lambda_1(v + \omega),\\
\sin\theta &= \sin i\,\mathrm{sn}\,u,\\
\phi - \Omega &= (\lambda_2\cos i/\lambda_1)\\
&\quad\times\int_0^u du/(1 - \sin^2 i\,\mathrm{sn}^2 u).
\end{aligned}
\tag{4}
$$

Here a, e, i, σ, ω, Ω are constants analogous to the usual elliptic elements, and

$$
\begin{aligned}
p &= a(1-e^2), \qquad n^2a^3 = \mu,\\
\epsilon &\equiv 6kc_1/\mu p,\\
\lambda_1^2 &\equiv 1 + \epsilon(c_2 + \cos^2 i),\\
\lambda_2^2 &\equiv 1 + \epsilon(c_2 - \sin^2 i),
\end{aligned}
\tag{5}
$$

the modulus κ of the Jacobian elliptic function sn being given by

$$\kappa^2 = (\epsilon/\lambda_1^2)\sin^2 i. \tag{6}$$

(In the original paper, with the results carried only to $O(k)$, the factor λ_1^2 in (6) was omitted.) In terms of the circular functions the last three of the equations of the orbit become:

$$
\begin{aligned}
\bar{\psi} &\equiv (1 + g_{21})(v + \omega),\\
\psi &= \bar{\psi} + \tfrac{1}{8}\kappa^2\sin 2\bar{\psi} + \cdots,\\
\sin\theta &= \sin i\sin\psi,\\
\phi &= \Omega + \tan^{-1}(\cos i\tan\psi) + g_{32}\bar{\psi} + \cdots,
\end{aligned}
\tag{7}
$$

where the constants g_{21} and g_{32} are related below to the fundamental frequencies of the motion.

The three fundamental angular frequencies have been given by

$$
\begin{aligned}
n_1 &= n,\\
n_2 &= n_1\lambda_1\pi/2K,\\
n_3 &= (n\lambda_2\cos i/K)\int_0^K du/(1 - \sin^2 i\,\mathrm{sn}^2 u),
\end{aligned}
\tag{8}
$$

where $K(\kappa)$ is the complete elliptical integral of the first kind, with the series expansion in κ^2,

$$K = \frac{\pi}{2}\left(1 + \frac{\kappa^2}{4} + \frac{9\kappa^4}{64} + \cdots\right). \tag{9}$$

The quantities

$$
\begin{aligned}
g_{21} &= n_2/n_1 - 1,\\
g_{32} &= n_3/n_2 - 1,
\end{aligned}
\tag{10}
$$

in view of (8), (9), (5), and (6) can be approximated to $O(k^2)$ by

$$
\begin{aligned}
g_{21} &= \lambda_1\left(1 - \frac{\kappa^2}{4} - \frac{5\kappa^4}{64}\right) - 1\\
&= \tfrac{1}{4}\epsilon(3\cos^2 i - 1 + 2c_2)\\
&\quad + \tfrac{1}{64}\epsilon^2[-5 + 8c_2 - 8c_2^2\\
&\quad + (18 - 24c_2)\cos^2 i - 21\cos^4 i]\\
g_{32} &= -\tfrac{1}{2}(\epsilon/\lambda_1^2)\cos i\\
&\quad - \tfrac{1}{16}\epsilon^2(5\cos i - 3\cos^3 i).
\end{aligned}
\tag{11}
$$

The orbit can be characterized as a pseudo-ellipse; the motion in r is identical with that in a Keplerian ellipse with $V_0 = -\mu/r$. The variables E, v, ψ are the analogues of the eccentric anomaly, the true anomaly, and the argument of latitude. If $c_i = 0$, then $\kappa = 0$, $g_{21} = g_{32} = 0$, $\lambda_1 = \lambda_2 = \mu = 1$, and the orbit degenerates into an ellipse.

If the c_i's assume the values

$$
\begin{aligned}
c_1 &= \frac{1}{p}, \qquad c_2 = \cos^2 i,\\
c_3 &= (3\cos^2 i - 1)\sqrt{1 - e^2}/4p^2,
\end{aligned}
\tag{12}
$$

the secular variations of $O(k)$ are zero. With the abbreviations

$$
\epsilon = \frac{6k}{\mu p^2}, \qquad P = \tfrac{1}{4}(1 - 3\cos^2 i), \qquad Q = \tfrac{1}{4}\sin^2 i,
\tag{13}
$$

the quantities λ_1, λ_2, μ, g_{21}, g_{32}, which involve the c_i's explicitly, become on the substitution from (12):

$$
\begin{aligned}
\lambda_1^2 &= 1 + 2\epsilon\cos^2 i, \qquad \lambda_2^2 = 1 + \epsilon\cos 2i,\\
\mu &= 1 + 6kP\sqrt{1 - e^2}/p^2,
\end{aligned}
\tag{14}
$$

to $O(k)$, and

$$g_{21} = \frac{\epsilon}{4}(5\cos^2 i - 1)$$

$$+\frac{\epsilon^2}{64}(-5+26\cos^2 i-53\cos^4 i), \quad (15)$$

$$g_{32}=-\frac{\epsilon}{2}\cos i+\frac{\epsilon^2}{16}(-5\cos i+19\cos^3 i),$$

to $O(k^2)$. The non-Keplerian orbit defined by the above choice of the c_i's will be designated as the "intermediary" and will be taken as the unperturbed orbit in the Delaunay theory. The main problem, defined by $k'=0$, will be solved first; the perturbations arising from the fourth spherical harmonic will be calculated in section 10.

3. *The old Hamiltonian.* The Delaunay variables for the intermediary orbit are:

$$\begin{aligned} L&=\sqrt{\mu a}, & l&=\int n dt+\sigma,\\ G&=L\sqrt{1-e^2}, & g&=\omega,\\ H&=\lambda_2 G\cos i, & h&=\Omega. \end{aligned} \quad (16)$$

Except for the appearance of the factor λ_2, which is nearly unity, they are identical in form with those for a Keplerian ellipse with $V_0=-\mu/r$. The Hamiltonian is

$$F=\mu^2/2L^2+R, \quad (17)$$

where $-\mu^2/2L^2$ is the undisturbed total energy, and the disturbing function R is to be expressed in terms of $L, G, H; l, g, h$. From (1) and (3)

$$\begin{aligned} R=&-2k[P_2(\sin\theta)/r^3\\ &-3c_1(\sin^2\theta-c_2)/2r^2-3c_3/r]. \end{aligned} \quad (18)$$

In the analysis the c_i's must be shown explicitly until all the necessary differentiations with respect to a, e, i have been performed. With the abbreviations (13), the replacement of $\sin\theta$ by $\sin i\sin\psi$, followed by the replacement of ψ by $\bar\psi+\frac{1}{2}\epsilon Q\sin 2\bar\psi+\cdots$, in accord with (7), converts the last expression into

$$\begin{aligned} R&=R_1+R_2+\cdots,\\ R_1&\equiv\frac{\mu\epsilon}{p}\left[\tfrac{1}{3}(3Q\cos 2\psi-P)\left(\frac{p}{r}\right)^3\right.\\ &\qquad+\tfrac{1}{2}c_1 p(4Q\sin^2\psi-c_2)\left(\frac{p}{r}\right)^2\\ &\qquad\left.+c_3p^2\left(\frac{p}{r}\right)\right],\\ R_2&\equiv\frac{\mu\epsilon^2}{p}c_1pQ^2\left[c_1p\left(\frac{p}{r}\right)^2-\left(\frac{p}{r}\right)^3\right]\sin^2 2\psi, \end{aligned} \quad (19)$$

where the subscripts of R indicate the order of magnitude, and ψ has been written for $\bar\psi$. A standard form for R as a double Fourier series in v and ψ can be derived by the replacement of p/r by $1+e\cos v$.

In the application of the Delaunay method it will be necessary to construct the secular and the long-period parts of R and of its derivatives R_L, R_G. Let ξ be the slowly-varying quantity

$$\xi\equiv\psi-v\cong g_{21}v+g, \quad (20)$$

that is the analogue of the argument of the pericenter g_0 in elliptic motion. If now a function $f(l,\xi)$ has the Fourier expansion

$$f=\sum_0^\infty A_{ij}\cos(il+2j\xi), \quad (21)$$

then its secular part $\bar f$ and its long-period part f^* are defined by

$$\bar f=A_{00},\quad f^*=\sum_0^\infty A_{0j}\cos 2j\xi,\quad j\neq 0. \quad (22)$$

The Fourier coefficients in (22) are given by the integrals

$$\begin{aligned} A_{00}&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f dl d\xi,\\ A_{0j}&=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\cos 2j\xi dl d\xi,\quad j\neq 0. \end{aligned} \quad (23)$$

The change of variables $l,\xi\to v,\xi$ may be carried out with the aid of the law of areas,

$$dl=(nr^2/G)dv, \quad (24)$$

which is rigorously valid only in the unperturbed orbit. However, the resulting error is of a higher order, so that, to $O(kf)$, (23) becomes

$$\begin{aligned} A_{00}&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(nr^2 f/G)dv d\xi,\\ A_{0j}&=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}(nr^2 f/G)\cos 2j\xi dv d\xi,\\ &\qquad j\neq 0. \end{aligned} \quad (25)$$

It follows immediately that the principal parts of A_{00} and A_{0j} are respectively the constant term and the coefficient of $\cos 2j\xi$, $j\neq 0$, in the expansion of $nr^2 f/G$ into a Fourier series in v and ξ. If such an expansion is available, the integration (25) is unnecessary, for A_{00} and A_{0j} can be found by direct inspection. The following results will be useful:

$$\overline{\left(\frac{p}{r}\right)^3} = \overline{\left(\frac{p}{r}\right)^2} = \overline{\left(\frac{p}{r}\right)}\sqrt{1-e^2} = (1-e^2)^{\frac{3}{2}}, \tag{26}$$

$$\overline{\cos 2\psi} = 0.$$

The secular part of R now follows by the substitution from (26) into (19):

$$\overline{R_1} = \frac{\mu\epsilon}{p}(1-e^2)^{\frac{3}{2}}[-\tfrac{1}{3}P + c_1p(Q - c_2/2) + c_3p^2/\sqrt{1-e^2}], \tag{27}$$

$$\overline{R_2} = \frac{\mu\epsilon^2}{2p}(1-e^2)^{\frac{3}{2}}Q^2c_1p(c_1p-1).$$

In terms of the Delaunay variables,

$$\overline{R_1} = \frac{k\mu^3}{L^3G^3}\left[-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2\lambda_2^2} + \frac{3c_1G^2}{2\mu}\left(1 - 2c_2 - \frac{H^2}{G^2\lambda_2^2}\right) + \frac{6c_3G^3L}{\mu^2}\right], \tag{28}$$

$$\overline{R_2} = \frac{9k^2c_1}{8L^3G^5}(c_1G^2 - 1)\left(1 - \frac{H^2}{G^2\lambda_2^2}\right)^2 + O(k^3),$$

with

$$\lambda_2^2 = 1 + 6kc_1[(c_2-1)G^{-2} + H^2G^{-4}], \tag{29}$$

in view of (5), (13), and (16). With $f = R$ in (25) observe that in the expression r^2R the terms involving ψ and r are of the form $\cos 2\psi/r$ and $\cos 4\psi/r$. Consequently, in the expansion

$$\begin{aligned} r^2R &= \sum A_{k_1k_2}\cos(k_1v + 2k_2\psi) \\ &= \sum A_{ij}\cos(iv + 2j\xi) \end{aligned} \tag{30}$$

$k_2 = 0, 1, 2$, and if $k_2 \neq 0$ then $|k_1| = 0, 1$. Since $i = k_1 + 2k_2$ and $j = k_2$, the inequality $j \neq 0$ implies $i \neq 0$ and $A_{0j} = 0$, so that

$$R^* = 0; \tag{31}$$

i.e., the disturbing function has no long-period part.

The derivatives of R_1 with respect to L and G, which will be needed later, are obtained with the aid of the relations

$$R_L = R_a a_L + R_e e_L, \quad R_G = R_e e_G + R_i i_G, \tag{32}$$

the literal subscripts indicating the arguments of partial differentiation. The quantities R_a, R_e, R_i to $O(k)$, and the Poisson brackets a_L, e_L, e_G, i_G given in the earlier paper, pp. 91–92, yield on the substitution into (32):

$$\begin{aligned}(r^2R_{1L}/G)_1 = \frac{\epsilon G}{eL}\Big\{&(P - 3Q\cos 2\psi)\left[e - \left(1 - \frac{5}{4}e^2\right)\cos v - e\cos 2v - \frac{e^2}{4}\cos 3v\right] \\ &- Q\sin 2\psi\left[\left(4 + \frac{e^2}{2}\right)\sin v + 3e\sin 2v + \frac{e^2}{2}\sin 3v\right] \\ &+ \tfrac{1}{2}c_1p(4Q\sin^2\psi - c_2)(-3e + 2\cos v + e\cos 2v) \\ &+ c_1pQ\sin 2\psi(4\sin v + e\sin 2v) + c_3p^2\left(\cos v - \frac{2er}{p}\right)\Big\}\end{aligned} \tag{33.1}$$

$$\begin{aligned}(r^2R_{1G}/G)_1 = \frac{\epsilon}{e}\Big\{&(P - 3Q\cos 2\psi)\left[e + (1 + \tfrac{3}{4}e^2)\cos v + e\cos 2v + \frac{e^2}{4}\cos 3v\right] \\ &+ Q\sin 2\psi\left[\left(4 + \frac{e^2}{2}\right)\sin v + 3e\sin 2v + \frac{e^2}{2}\sin 3v\right] - e\cos^2 i\sin^2\psi(1 + e\cos v) \\ &- \tfrac{1}{2}c_1p(4Q\sin^2\psi - c_2)(e + 2\cos v + e\cos 2v) - c_1pQ\sin 2\psi(4\sin v + e\sin 2v) \\ &+ c_1pe\cos^2 i\sin^2\psi - c_3p^2\cos v\Big\}.\end{aligned} \tag{33.2}$$

Since the terms $\cos(2\psi - 2v)$ cancel out in (33.1) and (33.2),

$$(R_{1L}^*)_1 = (R_{1G}^*)_1 = 0; \tag{34}$$

i.e. the derivatives of the disturbing function with respect to L and G have no long-period parts of $O(k)$. It must be noted, however, that the integration of a long-period term with respect to the time is accompanied by a reduction of the order of magnitude by unity. For this reason it is necessary to carry the analysis to $O(k^2)$ in order to obtain the long-period term to $O(k)$ in the final solution. Then the approximation $\psi_v = 1$ of the earlier paper must be replaced by

$\psi_v = 1 + g_{21} + g_{21}Q \cos 2\psi$. The term g_{21} yields long-period parts of $O(k^2)$:

$$(r^2R_{1L}^*/G)_2 = -\epsilon g_{21}Q(G/L)\cos 2\xi,$$
$$(r^2R_{1G}^*/G)_2 = +\epsilon g_{21}Q\cos 2\xi, \quad (35)$$

with $\xi = \psi - v$ and $g_{21} = O(\epsilon)$. The term $g_{21}Q \cos 2\xi$ contributes merely short-period terms of $O(k^2)$, which do not concern us here.

The treatment of g_{21} as a constant is justified by the circumstance that the dependence of g_{21} on G contributes only a pure periodic term of $O(k^2)$ to angle ψ. Indeed, this contribution is

$$\Delta\psi = \Delta g + (\delta g_{21})\psi = \Delta g + (g_{21})_G(\delta G)\psi$$
$$= (g_{21})_G \int \delta G d\psi,$$

inasmuch as, to the second order,

$$\Delta g = -\int \Delta R_G dt = -\int R_g (g_{21})_G \psi dt$$
$$= -(g_{21})_G \int \psi d(\delta G).$$

Since δG is purely periodic, it follows that the mixed secular term in the element g is cancelled in the perturbation of the position coordinates.

4. *A method of Delaunay*. In the old Hamiltonian, $F(L, G, H; l, g)$, the variable h is absent in consequence of the assumed axial symmetry; the time t is absent, having been incorporated into l. The conservation of the axial component of the angular momentum, $H =$ const., and the conservation of energy, $F =$ const., follow immediately from Hamilton's canonical equations. The problem is thus reduced to two degrees of freedom. We seek a transformation to new variables L', G', l', g' such that l', g' are absent in the new Hamiltonian F'. Then L' and G' will also be constants of the motion, and l', g' linear functions of the time. The secular variations can then be derived from F', and the periodic variations from the transformation function S.

In his Lunar Theory, Delaunay used a succession of canonical transformations to remove, one by one, the periodic terms of the Hamiltonian. The feasibility of a single transformation to accomplish the same purpose was suggested by von Zeipel (1916). Its first application to artificial satellites was made by Brouwer, who used the elliptic orbit, $c_i = 0$, as a first approximation. For any choice of the c_i's, the generating function $S(L', G'; l, g)$ of the desired canonical transformation satisfies the relations

$$L = S_l, \quad G = S_g,$$
$$l' = S_{L'}, \; g' = S_{G'}. \quad (36)$$

Since R is of $O(k)$, S must differ from the identity transformation,

$$S_0 = L'l + G'g + H'h, \quad (37)$$

also by a quantity of $O(k)$, and it must include terms S_i of short period, as well as terms S_i^* of long period. Accordingly, let

$$S = L'l + G'g + H'h + S_1 + S_1^* + \cdots,$$
$$S_i = O(k^i), \quad (38)$$

and, up to $O(k^2)$,

$$S_l = L' + S_{1l} + S_{1l}^* + S_{2l},$$
$$S_g = G' + S_{1g} + S_{1g}^* + S_{2g} + S_{2g}^*. \quad (39)$$

Since S^* contains l and g only in the combination $\xi = g_{21}v(l) + g$,

$$S_{1l}^* = g_{21}v_l S_{1g}^* = O(k^2),$$
$$S_{2l}^* = g_{21}v_l S_{2g}^* = O(k^3), \quad (40)$$

thus justifying the omission of S_{2l}^* in (39). The new Hamiltonian must satisfy the relation

$$F(L, G; l, g) + S_t = F'(L', G'). \quad (41)$$

In view of (36) and $S_t = 0$, (41) becomes

$$F(S_l, S_g; l, g) = F'(L', G'). \quad (42)$$

Now let

$$F = F_0 + F_1 + F_2 + \cdots; \quad F_i = O(k^i),$$
$$F' = F_0' + F_1' + F_2' + \cdots, \quad (43)$$

where we identify

$$F_0 = \mu^2/2L^2, \quad F_1 = R_1, \quad F_2 = R_2. \quad (44)$$

The substitution from (43) and (39) into (42), followed by an expansion of the left-hand member into a Taylor series about the "mixed" variables L', G', l, g, yields, up to $O(k^2)$:

$$F_0' + F_1' + F_2'$$
$$= F_0(L') + F_{0L}(S_{1l} + S_{1l}^* + S_{2l})$$
$$+\tfrac{1}{2}F_{0LL}S_{1l}^2 + F_1 + F_{1L}S_{1l}$$
$$+ F_{1G}(S_{1g} + S_{1g}^*) + F_2. \quad (45)$$

In view of (40) and the relations

$$F_{0L} = -n, \quad F_{0LL} = 3n/L, \quad (46)$$

separation of (45) by the orders of magnitude yields

$$F_0' = F_0 = \mu^2/2L'^2,$$
$$F_1' = F_1 - nS_{1l},$$
$$F_2' = F_2 + F_{1L}S_{1l} + F_{1G}S_{1g} + (3n/2L')S_{1l}^2 + (F_{1G} - g_{21}\dot{v})S_{1g}^* - nS_{2l}, \tag{47}$$

with the arguments L, G replaced by L', G'. The secular part of (47.2) is

$$F_1' = \overline{F_1} = \overline{R_1}, \tag{48}$$

since S_{1l} is purely periodic. Now S_1 can be determined from (47.2) as

$$S_1 = \frac{1}{n}\int (R_1 - \overline{R_1})dl = \int (r^2R_1/G)dv - \overline{R_1}l/n. \tag{49}$$

The secular part of (47.3) is

$$F_2' = \overline{F_2} + \overline{\Phi} = \overline{R_2} + \overline{\Phi}, \tag{50}$$

where $\overline{\Phi}$ is the secular part of Φ, defined by

$$\Phi = F_{1L}S_{1l} + F_{1G}S_{1g} + (3n/2L')S_{1l}^2. \tag{51}$$

The remaining terms in (47.3) contribute nothing to $\overline{\Phi}$, since $\dot{v}$ and F_{1G} have no long-period parts, while S^*_{1g} is purely long-periodic. Therefore the long-period part of (47.3) is

$$0 = (\overline{F_{1G} - g_{21}\dot{v}})S_{1g}^* + \Phi^*, \tag{52}$$

where Φ^* is the long-period part of Φ. The expression in parentheses equals $-ng_{21}$, since $\bar{\dot{v}} = n$, and since the vanishing of the first-order secular variations on the intermediary implies $\bar{F}_{1G} = 0$. Now let

$$S^*_1 = \sum a_j \sin 2j\xi, \qquad \Phi^* = \sum A_j \cos 2j\xi, \tag{53}$$

where $\xi = g_{21}v + g$ with L, G replaced by L', G' wherever they occur as arguments of v. Then S_1^* can be determined from

$$a_j = A_j/2jng_{21}, \tag{54}$$

A_j being the coefficient of $\cos 2j(\psi - v)$ in the Fourier expansion of $nr^2\Phi/G$ in ψ and v.

The new Hamiltonian F' now furnishes the secular variations through the Hamilton equations

$$\dot{l}' = -F_{L'}', \quad \dot{g}' = -F_{G'}', \quad \dot{h}' = -F_{H'}', \tag{55}$$

while the periodic variations follow from the equations

$$\begin{aligned} L &= S_l = L' + S_{1l}, \\ G &= S_g = G' + S_{1g} + S_{1g}^*, \\ H &= S_h = H', \\ l' &= S_{L'} = l + S_{1L'} + S_{1L'}^*, \\ g' &= S_{G'} = g + S_{1G'} + S_{1G'}^*, \\ h' &= S_{H'} = h + S_{1H'} + S_{1H'}^*, \end{aligned} \tag{56}$$

involving the derivatives of the generating function S. Explicit expressions for these variations will be developed in the following sections. In the process the values of the c_i's will be redetermined.

5. *The redetermination of the c_i's.* The disposable parameters are to be determined so as to remove the secular variations of $O(k)$. The latter are obtained by the differentiation of F_1', which is given by $\overline{R_1}$ in (28.1) with L, G replaced by L', G'. The derivatives of λ_2, which appear in this calculation, being of $O(k)$ and multiplied by k, may contribute only terms of $O(k^2)$. Accordingly, with λ_2 fixed,

$$\begin{aligned} \dot{l}_1' &= \frac{3k\mu^3}{L'^4G'^3}\left[-\tfrac{1}{2}+\tfrac{3}{2}\left(1-\frac{c_1G'^2}{\mu}\right)\left(\frac{H}{G'\lambda_2}\right)^2+\tfrac{3}{2}\frac{c_1G'^2}{\mu}(1-2c_2)+4c_3\frac{G'^3L'}{\mu^2}\right] \\ &= n\epsilon\sqrt{1-e^2}(-P+3M_2+2M_3), \\ \dot{g}_1' &= \frac{3k\mu^3}{L'^3G'^4}\left[-\tfrac{1}{2}+\tfrac{1}{2}\left(5-\frac{3c_1G'^2}{\mu}\right)\left(\frac{H}{G'\lambda_2}\right)^2+\frac{1}{2\mu}c_1G'^2(1-2c_2)\right] \\ &= \tfrac{1}{4}n\epsilon\left[5\cos^2 i-1+4c_1p\left(P-\frac{c_2}{2}\right)\right], \\ \dot{h}_1' &= \frac{3k\mu^3}{L'^3G'^4}\left(\frac{c_1G'^2}{\mu}-1\right)\left(\frac{H}{G'\lambda_2}\right) \\ &= \tfrac{1}{2}n\epsilon\cos i(c_1p-1), \end{aligned} \tag{57}$$

with the abbreviations

$$M_2 = c_1p(Q - c_2/2), \quad M_3 = c_3p^2/\sqrt{1 - e^2}, \tag{58}$$

and with a, e, i redefined so that

$$L' = \sqrt{\mu a}, \quad G' = L'\sqrt{1 - e^2}, \quad H = \lambda_2 G' \cos i. \tag{59}$$

For the vanishing of the first-order variations in (57) it is necessary and sufficient that the c_i's assume the values in (12) with a, e, i redefined as in (59). For $c_i = 0$ (57) reduces to the elliptic values:

$$\begin{aligned} \dot{l}_1' &= \tfrac{1}{4}n\epsilon(3\cos^2 i - 1)\sqrt{1 - e^2}, \\ \dot{g}_1' &= \tfrac{1}{4}n\epsilon(5\cos^2 i - 1), \\ \dot{h}_1' &= -\tfrac{1}{2}n\epsilon \cos i. \end{aligned} \tag{60}$$

6. *The new Hamiltonian.* The new Hamiltonian is

$$F' = \mu^2/2L'^2 + \overline{R_1} + \overline{R_2} + \Phi, \tag{61}$$

with R_1 and R_2 furnished by (28) and L, G replaced by L', G'. To evaluate Φ from the definition (51), expressions for $(R_{1L})_1$, $(R_{1G})_1$, S_{1l}, S_{1g} are required. The first two are furnished by (33); the last two will be now derived from S_1 in (49). First, with the aid of (19.1) and (27.1):

$$\begin{aligned} r^2R_1/G &= \epsilon G\Big[(Q\cos 2\psi - \tfrac{1}{3}P)\left(\frac{p}{r}\right) \\ &\quad + c_1p\left(2Q\sin^2\psi - \frac{c_2}{2}\right) + c_3p^2\left(\frac{r}{p}\right)\Big], \end{aligned} \tag{62}$$

$$\overline{R_1}/n = \epsilon G[-\tfrac{1}{3}P + c_1p(Q - c_2/2) + c_3p^2/\sqrt{1 - e^2}].$$

Then the replacement by $1 + e\cos v$ of p/r in (62.1) is followed by a conversion to a Fourier series in ψ and v. Prior to the integration of the resulting expression it is necessary to replace L and G by L', G' wherever they occur as arguments, implicitly and explicitly. In order to avoid "mixed" variables here, as well as in other algebraic operations throughout the analysis, a total replacement of all the old variables by the new will be made whenever L and G must be replaced by L', G'. This practice is justified by the fact that the resulting error is of an order higher than our tolerance. The integration of (62.1) is further facilitated by the use of the approximation $\psi = v + g$, with g constant to $O(k)$, and of the relation

$$\int_0^v \frac{r}{p}\sqrt{1 - e^2}\,dv = E = l + e\sin E, \tag{63}$$

which is rigorously valid only in the unperturbed motion. Again, the resulting errors are of a higher order, so that, to $O(k)$,

$$\begin{aligned} S_1 = \epsilon G\Big\{&-\tfrac{1}{3}P(v - l + e\sin v) \\ &+ \tfrac{1}{2}Q\left[\sin 2\psi + e\sin(2\psi - v) + \frac{e}{3}\sin(2\psi + v)\right] \\ &- \tfrac{1}{2}c_1pQ\sin 2\psi + c_1p\left(Q - \frac{c_2}{2}\right)(v - l) \\ &+ (c_3p^2/\sqrt{1 - e^2})\,e\sin E\Big\}. \end{aligned} \tag{64}$$

Now the differentiation of S_1 in (49) with respect to l, followed by the use of (19.1) and (62.2), yields S_{1l}; the differentiation of S_1 in (64) with respect to g, which enters only through $\psi = v + g$, yields S_{1g}:

$$\begin{aligned} S_{1l} &= \frac{1}{n}(R_1 - \overline{R_1}) \\ &= \frac{\mu\epsilon}{np}\Big[\tfrac{1}{3}(3Q\cos 2\psi - P)\left(\frac{p}{r}\right)^3 + \tfrac{1}{3}P(1 - e^2)^{3/2} \\ &\quad + \tfrac{1}{2}c_1p(4Q\sin^2\psi - c_2)\left(\frac{p}{r}\right)^2 \\ &\quad - c_1p(Q - c_2/2)(1 - e^2)^{\frac{3}{2}} \\ &\quad + c_3p^2\left(\frac{p}{r} - 1 + e^2\right)\Big], \end{aligned} \tag{65}$$

$$\begin{aligned} S_{1g} = \epsilon GQ\Big[&e\cos(2\psi - v) \\ &+ (1 - c_1p)\cos 2\psi + \frac{e}{3}\cos(2\psi + v)\Big]. \end{aligned}$$

The last term of (51) leads to

$$\begin{aligned} \frac{3n}{2L'}\overline{S_{1l}^2} &= \frac{3}{2nL'}\overline{(R_1 - \overline{R_1})^2} \\ &= \frac{3}{2nL'}(\overline{R_1^2} - \overline{R_1}^2), \end{aligned} \tag{66}$$

which is easily evaluated from R_1 in (19) with the aid of (26), and from $\overline{R_1}$ in (27). Therefore $\bar{\Phi}$ is the constant term in the Fourier expansion in v and ψ of the expression

$$\frac{nr^2}{G}(R_{1L}S_{1l} + R_{1G}S_{1g}) + \frac{3}{2nL}(\overline{R_1^2} - \bar{R}_1^2). \tag{67}$$

The multiplication of the double Fourier series for R_{1L}, R_{1G}, S_{1l}, S_{1g} is perfectly straightforward; after some laborious algebra, the second order portion F_2' of the new Hamiltonian is found as the sum $F_2' = \bar{R}_2 + \Phi$, with L and G replaced

by L', G':

$$F_2' = \frac{3}{32}\frac{k^2}{L'^3G'^7}\{-5+4x+5x^2$$
$$+y^2(10-24x-18x^2)+y^4(35+36x+5x^2)$$
$$+6c_1^2G'^4[3-8c_2+8c_2^2+y^2(8c_2-2)-y^4]$$
$$+24c_1G'^2[-1+2c_2+y^2(2-6c_2)-y^4]\}, \quad (68)$$

with the abbreviations

$$x = G'/L', \quad y = H/G'\lambda_2. \quad (69)$$

We have omitted in (68) the portion

$$\frac{18k^2}{L^3G^7}(M_2+M_3)(3M_2+M_3-2P)x, \quad (70)$$

inasmuch as it contributes nothing to the perturbations of the intermediary. Indeed, from (58), (12), and (13)

$$M_2 = -M_3 = P, \quad (71)$$

and with the last two factors of (70) vanishing, the first-order derivatives of the entire expression must also vanish.

7. *The secular variations.* Having removed the secular variations of $O(k)$, we proceed to calculate those of $O(k^2)$. Note that the quantity λ_2, entering F_1' through R_1 in (28), contributes nothing to the intermediary orbit since the derivatives of λ_2 are to be multiplied by the factor c_1p-1, which vanishes by (12). Furthermore, the derivatives of λ_2 are of $O(k)$; consequently λ_2 entering F_2' through $\bar{R}_2$ contributes only terms of $O(k^3)$. Accordingly, it is sufficient for our purpose to differentiate F_2' in (68), treating λ_2 as a constant, and then substitute the values of the c_i's from (12):

$$\dot{l}_2' = \frac{1}{384}n\epsilon^2x[-33+16x+25x^2$$
$$+y^2(138-96x-90x^2)$$
$$+y^4(-129+144x+25x^2)],$$
$$\dot{g}_2' = \frac{1}{384}n\epsilon^2[-101+24x+25x^2 \quad (72)$$
$$+y^2(462-192x-126x^2)$$
$$+y^4(-497+360x+45x^2)],$$
$$\dot{h}_2' = \frac{1}{96}n\epsilon^2y[-23+12x+9x^2$$
$$+y^2(43-36x-5x^2)],$$

with $x=\sqrt{1-e^2}$, $y=\cos i$, $\epsilon = 6k/G^4$, $n = L^{-3}$. For $c_i = 0$ (68) leads to the elliptic values,

$$(\dot{l}_2')_0 = \frac{\epsilon^2n}{384}x[-15+16x+25x^2$$
$$+y^2(30-96x-90x^2)$$
$$+y^4(105+144x+25x^2)],$$
$$(\dot{g}_2')_0 = \frac{\epsilon^2n}{384}[-35+24x+25x^2 \quad (73)$$
$$+y^2(90-192x-126x^2)$$
$$+y^4(385+360x+45x^2)],$$
$$(\dot{h}_2')_0 = -\frac{\epsilon^2n}{96}y[5-12x-9x^2$$
$$+y^2(35+36x+5x^2)],$$

in agreement with Brouwer.

8. *Variations of Short period.* From (50) the variations arising from S_1 are

$$\delta L = S_{1l}, \quad \delta G = S_{1g}, \quad \delta H = 0,$$
$$\delta l = -S_{1L'}, \quad \delta g = -S_{1G'}, \quad \delta h = -S_{1H'}. \quad (74)$$

The derivatives S_{1l} and S_{1g} are found directly by the substitution of the values of the c_i's from (12) into the expressions for S_{1l} and S_{1g} in (65); $S_{1L'}$ and $S_{1G'}$ are obtained by the partial differentiation of S_1 in (49). In view of the relations $n=\mu^2L^{-3}$, $dl = (nr^2/G)dv$, and the vanishing of $\bar{R}_L$, $\bar{R}_G$ on the intermediary, the latter operation yields:

$$S_{1L'} = 3S_1/L' + \int^v (r^2R_L/G)dv,$$
$$S_{1G'} = \int^v (r^2R_G/G)dv. \quad (75)$$

The values of the c_i's can now be substituted from (12) into the expressions for S_1, R_{1L}, R_{1G} in (64), (33), (35). With the retention of the long-period terms of $O(k^2)$:

$$S_1 = \epsilon G\{\tfrac{1}{3}P[2v-3E+l-e\sin v] + \tfrac{1}{2}Qe[\sin(2\psi - v)+\tfrac{1}{3}\sin(2\psi+v)]\},$$

$$r^2R_L/G = -(\epsilon G/eL)\left\{P\left[(-1+\sqrt{1-e^2}-\tfrac{5}{4}e^2)\cos v + \frac{e^2}{4}\cos 3v - 2e\left(\frac{r}{p}\sqrt{1-e^2}-1\right)\right]\right.$$
$$+\frac{Q}{2}\left[-\frac{e^2}{4}\cos(2\psi-3v) + 2eg_{21}\cos(2\psi-2v) + (-1+\tfrac{17}{4}e^2)\cos(2\psi - v)\right.$$

$$+ (-1 + \tfrac{13}{4}e^2)\cos(2\psi + v) - 4e\cos(2\psi + 2v) - \frac{5e^2}{4}\cos(2\psi + 3v)\Big]\Big\},$$

$$r^2R_G/G = -(\epsilon/e)\Big\{P\Big[(1 - \sqrt{1-e^2} - \tfrac{3}{4}e^2)\cos v - \frac{e^2}{4}\cos 3v\Big] \tag{76}$$

$$+\frac{Q}{2}\Big[\frac{e^2}{4}\cos(2\psi - 3v) - 2eg_{21}\cos(2\psi - 2v) + (1 + \tfrac{7}{4}e^2)\cos(2\psi - v) + 4e\cos 2\psi$$

$$+ (1 + \frac{11}{4}e^2)\cos(2\psi + v) + 4e\cos(2\psi + 2v) + \frac{5e^2}{4}\cos(2\psi + 3v)\Big]$$

$$+\frac{e^2}{2}\cos^2 i\,[\cos v - \tfrac{1}{2}\cos(2\psi - v) - \tfrac{1}{2}\cos(2\psi + v)]\Big\}.$$

The integration in (75) is facilitated by the use of the relation (63) and the approximations $\psi = v + g$ for the short-period terms and $\psi - v = g_{21}v + g$ for the long-period terms. The resulting error in δl and δg arising from all sources will be of $O(k^2)$. Finally, S_{1H} is obtained by the differentiation of S_1 in (64) with respect to H, which enters only through i in $P(i)$ and $Q(i)$, with $i_H = G^{-1}\cot i$, followed by the replacement of c_1p by unity. With the abbreviation

$$c = \frac{1}{3e}[(1-e^2)^{\frac{3}{2}} - 3(1-e^2)^{\frac{1}{2}} + 2] = \tfrac{1}{3}(x+2)(1-x)^{\frac{3}{2}}(1+x)^{-\frac{1}{2}} \tag{77}$$

the results are collected below:

$$\delta \log L = \epsilon e/(1-e^2)\Big\{P\Big[c + \Big(1 - \sqrt{1-e^2} - \frac{e^2}{4}\Big)\cos v - \frac{e^2}{12}\cos 3v\Big]$$

$$+ Q\cos 2\psi\Big[e + (1 + \tfrac{3}{4}e^2)\cos v + e\cos 2v + \frac{e^2}{4}\cos 3v\Big]\Big\},$$

$$\delta \log G = \epsilon e Q\,[\cos(2\psi - v) + \tfrac{1}{3}\cos(2\psi + v)],$$

$$\delta l = (\epsilon\sqrt{1-e^2}/e)\Big\{P\Big[(-1 + \sqrt{1-e^2} - \tfrac{1}{4}e^2)\sin v + \frac{e^2}{12}\sin 3v + e^2\sin E\Big]$$

$$+\frac{Q}{2}\Big[\frac{e^2}{4}\sin(2\psi - 3v) + e\sin(2\psi - 2v) + (-1 + \tfrac{5}{4}e^2)\sin(2\psi - v)$$

$$+ (-\tfrac{1}{3} + \tfrac{1}{12}e^2)\sin(2\psi + v) - e\sin(2\psi + 2v) - \frac{e^2}{4}\sin(2\psi + 3v)\Big]\Big\}, \tag{78}$$

$$\delta g = (\epsilon/e)\Big\{P\Big[(1 - \sqrt{1-e^2} - \tfrac{3}{4}e^2)\sin v - \frac{e^2}{12}\sin 3v\Big]$$

$$+\frac{Q}{2}\Big[-\frac{e^2}{4}\sin(2\psi - 3v) - e\sin(2\psi - 2v) + (1 + \tfrac{7}{4}e^2)\sin(2\psi - v) + 2e\sin 2\psi$$

$$+ (\tfrac{1}{3} + \tfrac{11}{12}e^2)\sin(2\psi + v) + e\sin(2\psi + 2v) + \frac{e^2}{4}\sin(2\psi + 3v)\Big]$$

$$+\frac{e^2}{2}\cos^2 i\,[\sin v - \tfrac{1}{2}\sin(2\psi - v) - \tfrac{1}{6}\sin(2\psi + v)]\Big\},$$

$$\delta h = \tfrac{1}{2}\epsilon e\cos i\,[-\sin v + \tfrac{1}{2}\sin(2\psi - v) + \tfrac{1}{6}\sin(2\psi + v)].$$

In addition to non-singular long-period terms in (78), there also appear in the solution terms with the singular factor $(5\cos^2 i - 1)^{-1}$, arising from S^*_1. Such terms will be considered in the next section.

9. *The long-period term.* From (56) the variations of long-period arising from S_1^* are, to $O(k)$,

$$\delta L^* = 0, \qquad \delta G^* = S_{1g}^*, \qquad \delta H^* = 0,$$
$$\delta l^* = -S_{1L'}^*,\ \ \delta g^* = -S_{1G'}^*,\ \ \delta h^* = -S_{1H}^*. \tag{79}$$

To obtain S_1^* from (53) and (54) it is necessary first to evaluate the long-period part of Φ from the definition (51). This evaluation is equivalent to the determination of the coefficients A_j of $\cos 2j(\psi - v)$ in the products of double Fourier series. In the result,

$$\Phi^* = A_1 \cos 2\xi$$
$$= \frac{1}{192} n\epsilon^2 G' e^2 \sin^2 i\,(1-15\cos^2 i)\cos 2\xi$$
$$A_2 = A_3 = \cdots = 0, \tag{80}$$

curiously, all the terms involving the c_i's explicitly and all the higher harmonics of 2ξ have vanished. The divisor ng_{21}, occurring in (54), is seen from (15) to be

$$ng_{21} = \tfrac{1}{4}\epsilon n(5\cos^2 i - 1) = \dot{g}_0' \tag{81}$$

to $O(k)$. This expression is identical with the secular rate of the advance of the argument of the pericenter g_0' in a perturbed ellipse, $c_i = 0$. That the trigonometric argument $\xi = g_{21}v + g$ is the analogue of g_0 follows from the equality of their secular rates and, to $O(1)$, from the equality of their initial values. Now S_1^* can be written

$$S_1^* = \frac{1}{96}\epsilon G'(1-x^2)(1-y^2)(1-15y^2)\times(5y^2-1)^{-1}\sin 2\xi$$
$$= \frac{k}{16G'^3}\left(1-\frac{G'^2}{L'^2}\right)\left[-1+11\frac{H^2}{G'^2}-40\frac{H^4}{G'^4}\left(5\frac{H^2}{G'^2}-1\right)^{-1}\right]\sin 2(g_{21}v+g). \tag{82}$$

Finally, the differentiation of S_1^*, followed by the restoration of ϵ, e, y, ξ in the right-hand members, and the use of the abbreviations

$$\alpha = \frac{1}{48}(1-y^2)(1-15y^2)(5y^2-1)^{-1},$$
$$\beta = \frac{1}{2}y\alpha_y = \frac{1}{48}y^2[3+8(5y^2-1)^{-2}],$$
$$\gamma = \beta/y, \tag{83}$$

yields:

$$\delta \log G^* = \epsilon e^2\alpha\cos 2\xi,$$
$$\delta l^* = -\epsilon(1-e^2)^{3/2}\alpha\sin 2\xi,$$
$$\delta g^* = \epsilon\left[\left(1+\frac{e^2}{2}\right)\alpha + e^2\beta\right]\sin 2\xi,$$
$$\delta h^* = -\epsilon e^2\gamma\sin 2\xi. \tag{84}$$

It is noteworthy that the coefficients of the long-period terms, being completely independent of the c_i's, are identical with what they would be in the elliptic approximation $c_i = 0$; indeed, (82) and (84) are in perfect agreement with Brouwer. Incidentally, (84.2) corrects the author's previously published erroneous value of the long-period term in the mean anomaly [1958].

The anomalistic and the draconic frequencies, n_1 and n_2, become equal at $i^* = \tan^{-1} 2 = 63°4$, giving rise to the phenomenon of "resonance," with singularities appearing as poles of orders one and two. The long-period is

$$P = \pi/ng_{21} = 4\pi/n\epsilon(5\cos^2 i - 1), \tag{85}$$

with

$$\min P = \pi/6k \tag{86}$$

corresponding to $a = 1$, $e = 0$, $i = 0$. For the earth with $k = 0.545 \times 10^{-3}$ and the canonical unit of time 806.8 sec., min $P = 8.9$ days.

10. *The effect of the fourth harmonic.* The contribution of the fourth spherical harmonic in the potential to the disturbing function is

$$\Delta R = -k' P_4(\sin\theta)/r^5. \tag{87}$$

From the definition

$$P_4(\sin\theta) = \tfrac{1}{8}(3 - 30\sin^2\theta + 35\sin^4\theta) \tag{88}$$

and the relation $\sin\theta = \sin i \sin\psi$ there follows:

$$P_4(\sin\theta) = \tfrac{3}{8}P_4(\cos i)$$
$$+\frac{5}{16}\sin^2 i\,(6-7\sin^2 i)\cos 2\psi$$
$$+\frac{35}{64}\sin^4 i\cos 4\psi. \tag{89}$$

Upon the replacement of $P_4(\sin\theta)$ and $(p/r)^5$ by their secular parts,

$$\bar{P}_4(\sin\theta) = \tfrac{3}{8}P_4(\cos i),$$
$$\overline{(p/r)^5} = (1-e^2)^{\frac{3}{2}}\left(1+\frac{3e^2}{2}\right), \tag{90}$$

the secular part of ΔR appears as

$$\Delta\bar{R} = -\frac{3}{8}\frac{k'}{p^5}(1-e^2)^{\frac{3}{2}}\left(1+\frac{3e^2}{2}\right)P_4(\cos i) = -\frac{3}{128}\frac{k'}{L^3G^7}\left(5-3\frac{G^2}{L^2}\right)\left(3-30\frac{H^2}{G^2}+35\frac{H^4}{G^4}\right), \tag{91}$$

and the contribution to the new Hamiltonian is

$$\Delta F' = \Delta \overline{R}, \tag{92}$$

with L, G replaced by L', G'. From $\Delta F'$ are derived the secular variations, which can be written in the form:

$$\begin{aligned}
\Delta \dot{l}' &= -\tfrac{3}{16} n\epsilon' x(1 - x^2)(3 - 30y^2 + 35y^4),\\
\Delta \dot{g}' &= \tfrac{1}{16} n\epsilon' [-21 + 270y^2 - 385y^4\\
&\qquad + x^2(9 - 126y^2 + 189y^4)],\\
\Delta \dot{h}' &= \tfrac{1}{4} n\epsilon'(5 - 3x^2)y(7y^2 - 3),
\end{aligned} \tag{93}$$

with the use of the abbreviations $x = \sqrt{1 - e^2}$, $y = \cos i$, and

$$\epsilon' \equiv 15k'/8p^4. \tag{94}$$

In view of (87) and (89), the discussion of section 3 makes it clear that the long-period part of ΔR arises only from the expression $\cos 2\psi/r^3$. Indeed,

$$\Delta R^* = \tfrac{1}{8} n\epsilon' G e^2 \sin^2 i\,(1 - 7\cos^2 i)\cos 2\xi, \tag{95}$$

with $\xi = \psi - v = g_{21}v + g$. This expression for ΔR^* appears as an additional term in the right-hand member of (52); the contribution to S_1^* is therefore

$$\begin{aligned}
\Delta S_1^* &= \frac{1}{4}\left(\frac{\epsilon'}{\epsilon}\right) G' e^2 \sin^2 i\,(1 - 7\cos^2 i)\,(5\cos^2 i - 1)^{-1}\sin 2\xi\\
&= \frac{1}{4}\epsilon_1\rho\frac{1}{G'^3}\left(1 - \frac{G'^2}{L'^2}\right)\left(1 - 8\frac{H^2}{G'^2} + 7\frac{H^4}{G'^4}\right)\left(5\frac{H^2}{G'^2} - 1\right)^{-1}\sin 2\,(g_{21}v + g),
\end{aligned} \tag{96}$$

with the constants ϵ_1 and ρ defined by

$$\epsilon_1 = \epsilon G'^4 = 6k, \quad \rho \equiv \epsilon'/\epsilon^2 = 5k'/96k^2. \tag{97}$$

From (96) are derived the long-period variations. With the use of the abbreviations

$$\begin{aligned}
\alpha' &= \tfrac{1}{2}(1 - y^2)(1 - 7y^2)(5y^2 - 1)^{-1},\\
\beta' &= \tfrac{1}{2} y\alpha'_y = \tfrac{1}{10} y^2 [7 + 8(5y^2 - 1)^{-2}],\\
\gamma' &= \beta'/y,
\end{aligned} \tag{98}$$

they can be written, analogously to (84),

$$\begin{aligned}
\delta \log G^* &= \epsilon\rho e^2 \alpha' \cos 2\xi,\\
\delta l^* &= -\epsilon\rho(1 - e^2)^{3/2}\alpha' \sin 2\xi,\\
\delta g^* &= \epsilon\rho\left[\left(1 + \frac{e^2}{2}\right)\alpha' + e^2\beta'\right]\sin 2\xi,\\
\delta h^* &= -\epsilon\rho e^2 \gamma' \sin 2\xi.
\end{aligned} \tag{99}$$

The short-period variations arising from the fourth harmonic are of $O(k^2)$ and can be neglected in our analysis.

11. *Computational procedure.* The secular variations of $O(k^2)$ can be incorporated into the orbit equations (7) by replacing n, g_{21}, g_{32} by n', g_{21}', g_{32}' defined as follows:

$$\begin{aligned}
n' &= n + \dot{l}'_2 + \Delta \dot{l}',\\
g_{21}' &= g_{21} + (\dot{g}_2' + \Delta \dot{g}')/n,\\
g_{32}' &= g_{32} + (\dot{h}_2' + \Delta \dot{h}')/n.
\end{aligned} \tag{100}$$

The substitution from (15), (72), (93) into (100) yields the values that appear in the summary below.

It is convenient to apply the periodic variations in L, G, l, g directly to the coordinates, with the aid of the first-order differential relations:

$$\begin{aligned}
\delta r &= r_a\delta a + r_e\delta e + r_l\delta l,\\
\delta\psi &= \delta v + \delta g = v_e\delta e + v_l\delta l + \delta g\\
&= (\delta g + \delta l/\sqrt{1 - e^2}) + v_e\delta e\\
&\quad - [(1 - v_l\sqrt{1 - e^2})/e]e\delta l/\sqrt{1 - e^2}
\end{aligned} \tag{101}$$

with the partial derivatives

$$\begin{aligned}
r_a &= r/a, \quad r_e = -a\cos v,\\
r_l &= ae\sin v/\sqrt{1 - e^2},\\
v_e &= (2 + e\cos v)\sin v/(1 - e^2),\\
v_l &= a^2\sqrt{1 - e^2}/r^2.
\end{aligned} \tag{102}$$

It is now clear that the singularity in δl and δg in (78) at $e = 0$ has been removed. The variations δa, δe, δi are related to δL and δG by

$$\begin{aligned}
\delta \log a &= 2\delta \log L,\\
\delta e &= -[(1 - e^2)/e]\delta \log (G/L),\\
\delta i &= \cot i\,\delta \log G,
\end{aligned} \tag{103}$$

which follows from the definitions (16). Finally, explicit expressions are derived for δr, $\delta\psi$, δi by the substitution from (78), (84), (99) into (101) and (103), followed by the substitution from (103) and (102) into (101). The results appear in the summary below, with the terms arising from S_1^* marked by an asterisk.

SUMMARY

A. The data (for a satellite of the earth)

1) the geophysical constants:

$$R = 6378.4 \text{ km.}, \quad n_0 = 1.23943 \times 10^{-3} \text{ rad./sec.}$$
$$J_2 = 2k = 1.083 \times 10^{-3},$$
$$J_4 = k' = -1.2 \times 10^{-6}$$

2) the elements $a, e, i, \sigma, \omega, \Omega$

B. Auxiliary constants:

$$x = \sqrt{1 - e^2} \qquad y = \cos i,$$
$$a_2 = \tfrac{1}{3}(1 - x) \qquad a_1 = a_2\sqrt{\frac{1 - x}{1 + x}}$$
$$P = \tfrac{1}{4}(1 - 3y^2), \quad Q = \tfrac{1}{4}(1 - y^2),$$
$$b = 1 + J_4/J_2^2$$
$$b_0 = 1 - \tfrac{7}{4}b$$
$$m = \tfrac{1}{12}b_0 - \tfrac{1}{24}b(5y^2 - 1)^{-1}$$
$$\alpha = 4mQ$$
$$\gamma = y[-\tfrac{1}{12}b_0 + \tfrac{1}{6}b(5y^2 - 1)^{-2}]$$
$$\beta = \gamma y$$
$$24\rho = 5(b - 1)$$
$$p = ax^2, \qquad \mu = 1 + 3J_2xP/p^2,$$
$$\epsilon = 3J_2/\mu p^2, \quad n = \mu^{1/2}a^{-3/2}n_0.$$

C. The secular effects:

$$n' = n\left\{1 + \frac{\epsilon^2 x}{384}[-33 + 16x + 25x^2 + y^2(138 - 96x - 90x^2) + y^4(-129 + 144x + 25x^2) - 72\rho(1 - x^2)(3 - 30y^2 + 35y^4)]\right\},$$

$$g_{21}' = \tfrac{1}{4}\epsilon(5y^2 - 1) + \frac{\epsilon^2}{384}\{-131 + 24x + 25x^2 + y^2(618 - 192x - 126x^2) + y^4(-815 + 360x + 45x^2) + 24\rho[-21 + 270y^2 - 385y^4 + x^2(9 - 126y^2 + 189y^4)]\},$$

$$g_{32}' = -\tfrac{1}{2}\epsilon y + \frac{\epsilon^2}{96}y[-53 + 12x + 9x^2 + y^2(157 - 36x - 5x^2) + 24\rho(5 - 3x^2)(7y^2 - 3)].$$

D. The equations of the orbit

$$l = n't + \sigma,$$
$$E = l + e \sin E,$$
$$\tan\frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}}\tan\frac{E}{2},$$
$$r = a(1 - e\cos E),$$
$$\psi = (1 + g_{21}')(v + \omega),$$
$$\delta r/r = \epsilon\left\{-P(a_1\cos v + a_2\cos 2v) + Q\left[\frac{e}{6}\cos(2\psi - v) + \tfrac{1}{3}\cos 2\psi + \frac{e}{6}\cos(2\psi + v)\right]\right\},$$
$$\delta r^*/r = \epsilon\alpha\left[\frac{e^2}{2}\cos(2\psi - 2v) + e\cos(2\psi - v) + \frac{e^2}{2}\cos 2\psi\right],$$
$$r' = r + \delta r + \delta r^*,$$
$$\delta\Omega = \tfrac{1}{2}\epsilon e\cos i\,[-\sin v + \tfrac{1}{2}\sin(2\psi - v) + \tfrac{1}{6}\sin(2\psi + v)],$$
$$\delta\Omega^* = -\epsilon e^2\gamma\sin(2\psi - 2v),$$
$$\Omega' = \Omega + \delta\Omega + \delta\Omega^*,$$
$$\delta\psi = \epsilon\{P(2a_1\sin v + a_2\sin 2v) + Q[\tfrac{2}{3}e\sin(2\psi - v) + \tfrac{1}{6}\sin 2\psi]\} - \cos i\,\delta\Omega,$$
$$\delta\psi^* = \epsilon\left[e^2\beta\sin(2\psi - 2v) - 2e\alpha\sin(2\psi - v) - \frac{e^2\alpha}{2}\sin 2\psi\right],$$
$$\psi' = \psi + \delta\psi + \delta\psi^*,$$
$$\delta i = \frac{\epsilon e}{8}\sin 2i\,[\cos(2\psi - v) + \tfrac{1}{3}\cos(2\psi + v)],$$
$$\delta i^* = \tfrac{1}{2}\epsilon m e^2\sin 2i\cos(2\psi - 2v),$$
$$i' = i + \delta i + \delta i^*,$$
$$\sin\theta' = \sin i'\sin\psi',$$
$$\phi' = \Omega' + \tan^{-1}(\cos i'\tan\psi') + g_{32}'\psi'.$$

The position of the satellite is furnished by the perturbed spherical coordinates r', θ', ϕ'. The equatorial radius R is the unit of distance; for computational convenience the time t is here expressed in seconds; the term $\frac{1}{8}\kappa^2 \sin 2\psi$, be-

longing to the unperturbed orbit in (7), has been incorporated into $\delta\psi$. The calculation of tangents could be avoided by the use of the alternate relations

$$\sin v = \frac{a\sqrt{1-e^2}}{r}\sin E, \quad \cos v = \frac{a}{r}(\cos E - e)$$

$$\phi' = \Omega' + \cos^{-1}(\cos\psi'/\cos\theta') + g_{32}'\psi'.$$

It is noteworthy that the solution appears in closed form; there is a singularity at $i = i_*$, but none at $e = 0$ or $i = 0$.

The constants J_i, introduced by King-Hele and Merson (1959), are the coefficients of the spherical harmonics in the potential V. They are related to the Jeffreys constants and to our k and k' by

$$J_0 = -1, \quad J_2 = \tfrac{2}{3}J = 2k,$$
$$J_4 = -\tfrac{8}{35}D = k'.$$

12. *The validity of the solution.* The solution is valid under the following restrictions: 1) $e < 1$, 2) $|t| < \tau$, where τ is a bound beyond which the omission of the secular variations of $O(k^3)$ can no longer be justified, 3) $|i - i_*| \geq w$, where the bound w is the half-width of the "resonance" region in the neighborhood of the singularity at i_*. In particular, w will be so defined that outside the region

$$|\delta\psi^*| \leq \epsilon_1 \equiv \epsilon p^2, \tag{104}$$

and consequently all the long-period terms can be treated like quantities of the first order. If $|i - i_*| \ll 1$, the formulae of the preceding section furnish the principal part of the bound on $|\delta\psi^*|$,

$$|\delta\psi^*| \leq \epsilon|2e\alpha + e^2\beta|. \tag{105}$$

With the use of the abbreviations

$$C = \frac{1}{120}|b|, \tag{106}$$

$$\Delta = \tfrac{1}{4}|5\cos^2 i - 1| \simeq |i - i_*|,$$

the dominant terms of the asymptotic series for α and β as $\cos^2 i \to \frac{1}{5}$ are

$$|\alpha| = C/\Delta, \quad |\beta| = C/4\Delta^2, \tag{107}$$

In order to assure the inequality (104) it is therefore sufficient that

$$\Delta/e \geq (C + \sqrt{C^2 + \tfrac{1}{4}Cp^2})/p^2. \tag{108}$$

If it is now assumed that

$$b = O(1), \tag{109}$$

then (108) can be approximated by $\Delta/e \geq \sqrt{C}/2p$, so that w can be chosen as

$$w = \frac{e}{2p}\sqrt{C}. \tag{110}$$

In particular, for an artificial satellite of the earth the relation (109) is certainly satisfied; furthermore, the launching conditions generally imply $p = 1 + e$. Then (110) becomes approximately

$$w = e|b|^{\frac{1}{2}}/20(1+e) \tag{111}$$

with max $w = 1.^\circ 4|b|^{\frac{1}{2}}$ corresponding to $e = 1$. The singularity at i_* dissappears when $w = 0$. It is noteworthy that this occurs when $e = 0$ and also when $b = 0$; the latter is characteristic of the Vinti potential (1959), with $J_4 + J_2^2 = 0$.

In a separate paper it will be shown how a non-singular solution valid for all i can be constructed by the retention of g' in the new Hamiltonian.

13. *Comparisons and checks.* Checks of the theory are provided by a comparison with the Brouwer solution of the main problem. Three such checks have been carried out, and perfect agreement obtained as far as the analysis went:

1) The amplitudes of the long-period variations are identical with those of an ellipse, $c_i = 0$.

2) Secular variations of $O(k)$ and of $O(k^2)$ reduce to those of an ellipse when the three disposable parameters c_i are set equal to zero.

3) Since the perturbed intermediary must be identical with the perturbed ellipse, the three natural frequencies must be the same in both orbits. These frequencies are:

$$\begin{aligned} n_1' &= \dot{l}' = n_1 + \dot{l}_1' + \dot{l}_2', \\ n_2' &= \dot{\bar\psi} = (1 + g_{21})(n_1' + \dot{g}_1' + \dot{g}_2'), \\ n_3' &= \dot{\bar\phi} = (1 + g_{32})n_2' + \dot{h}_1' + \dot{h}_2'. \end{aligned} \tag{112}$$

Proceeding from the facts that the bound of θ and the total energy F' must be the same in the two orbits, it can be shown that the constants L', G' in the two orbit are correlated by

$$\begin{aligned} \Delta G'/G_0' &= \epsilon P, \\ L'/L_0' &= \mu[1 - \tfrac{1}{3}\Delta(\epsilon PG/L) + L^2\Delta F_2'], \end{aligned} \tag{113}$$

where $\Delta G' = G' - G_0'$, etc.; and the subscript 0 refers to Brouwer's perturbed ellipse. Since $\cos i = H/\lambda_2 G'$, and $n = \mu^2/L^3$, it follows from the definitions of λ_2, P, ϵ, and from (68) that

$$\Delta i = -\frac{\epsilon}{4}\sin i \cos i,$$

$$\Delta\epsilon = -4\epsilon^2 P,$$

$$\Delta(\epsilon PG/L) = \tfrac{3}{16}\epsilon^2\sqrt{1-e^2}(1-4\cos^2 i+7\cos^4 i),$$

$$\frac{\Delta n}{n_0} = -(\epsilon PG/L)_0 + \frac{3}{64}\epsilon^2 x(1-6y^2+13y^4) = \mu^{-1} + O(k^2). \qquad (114)$$

With the aid of these results and the expressions for g_{21}, g_{32}, $\dot{l}'$, $\dot{g}'$, $\dot{h}'$, it has been established that $\Delta n_1 = \Delta n_2 = \Delta n_3 = 0$. Indeed, the anomalistic, the draconic, and the sidereal frequencies of the two orbits are respectively equal, as they must be.

The relative size of the variations in the two orbits is of interest. Not only do the secular variations of $O(k)$ vanish on the intermediary, but the short-period variations δL and δG vanish with the eccentricity e. The analysis of section 4 shows that the expressions for F_2', F_3', etc., involve the powers of δL and δG. It is therefore clear that the smallness of the latter quantities generally assures the smallness of secular variations of $O(k^2)$ and higher. Indeed, for a typical orbit with $e = 0$, $i = 45°$ the solution of the main problem yields the values

$$\dot{g}_2' = -\frac{1}{128}\epsilon^2 n, \quad \dot{h}_2' = -\frac{\sqrt{2}}{192}\epsilon^2 n,$$

which are only 3% and 6% respectively of the corresponding values

$$(\dot{g}_2')_0 = \frac{185}{768}\epsilon^2 n, \quad (\dot{h}_2')_0 = -\frac{17\sqrt{2}}{192}\epsilon^2 n$$

on the ellipse $c_i = 0$. The relative smallness of secular variations of $O(k^3)$ on our intermediary should guarantee the superior accuracy in the calculated position. This advantage is achieved without any sacrifice in the ease of orbit calculation.

14. *The effect of the third harmonic.* Our assumption of equatorial symmetry can be removed by the inclusion of the odd spherical harmonics in the potential V. The effect of the third harmonic will be calculated here by the method of section 10:

$$\Delta R = -k_3 P_3(\sin\theta)/r^4$$

$$P_3(\sin\theta) = \tfrac{1}{2}(5\sin^3\theta - 3\sin\theta)$$
$$= (\tfrac{15}{8}\sin^3 i - \tfrac{3}{2}\sin i)\sin\psi - \tfrac{5}{8}\sin^3 i \sin 3\psi$$

$$\Delta\bar{R} = 0$$

If $k_3 = O(k^2)$, only the long-period term need be considered, which arises only from the expression $\sin\psi/r^4$:

$$\Delta R^* = +\tfrac{3}{8}nk_3G^{-5}e\sin i\,(5\cos^2 i - 1)\sin\xi,$$

$$\Delta S_1^* = +\tfrac{1}{2}(k_3/k_2)G'^{-1}e\sin i\cos\xi,$$
$$= \tfrac{1}{2}(k_3/k_2)(1/G'^2 - 1/L'^2) \times (1 - H^2/G'^2)^{\frac{1}{2}}\cos\xi,$$

$$k_2 \equiv 2k, \quad \xi = \psi - v \simeq g_{21}v + g$$

With the use of the abbreviation

$$\epsilon_3 = k_3/2k_2G^2$$

the resulting variations in the Delaunay elements can be written:

$$\delta\log G^* = +\epsilon_3 e\sin i\sin\xi,$$

$$e\delta l^* = +\epsilon_3(1-e^2)^{\frac{1}{2}}\sin i\cos\xi,$$

$$\delta g^* = -\epsilon_3\left(\frac{\sin i}{e} - e\frac{\cos^2 i}{\sin i}\right)\cos\xi,$$

$$\delta h^* = -\epsilon_3 e\cot i\cos\xi.$$

With the end of (101), (102), and (103) of section 11, there follows:

$$\delta r^* = +\epsilon_3 G^2\sin\psi,$$

$$\delta\psi^* = +\epsilon_3\left[\frac{e}{\sin i} + (2\cos v - e\sin^2 v)\sin i\right] \times \cos\xi,$$

$$\delta i^* = +\epsilon_3 e\cos i\sin\xi,$$

$$i' = i + \delta i^*.$$

The remaining singularities in δh^* and $\delta\psi^*$ at $i = 0$ are removed in the variations of the coordinates θ and φ:

$$\sin\theta' = \sin i'(\sin\psi + \cos\psi\delta\psi^*)$$
$$= \sin i'\sin\psi + \epsilon_3\cos\xi\cos\psi \times [e + (2\cos v - e\sin^2 v)\sin^2 i]$$

$$\cos^2\theta\delta\varphi^* = -\epsilon_3\left[\frac{e}{2}(\sin^2 v - \sin^2\psi) - \cos v\right] \times \sin 2i\cos\xi$$

The variations δr, δi, $\delta \sin\theta$, $\delta\varphi$ can be readily incorporated into the computational scheme of section 11.

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THE MOTION OF A CLOSE EARTH SATELLITE

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Abstract. In the present paper perturbations of six orbital elements of a close earth satellite moving in the gravitational field of the earth without air-resistance are derived as functions of mean orbital elements and time. No assumptions are made about the order of magnitude of eccentricity and inclination. However, it is assumed that the density distribution of the earth is symmetrical with respect to the axis of rotation, that the coefficient of the second harmonic of the potential is a small quantity of the first order and that those of the third and the fourth harmonics are of the second order. The results include periodic perturbations of the first order and secular perturbations up to the second order.

However, the solutions have some singularities for an orbit whose eccentricity or inclination is smaller than a quantity of the first order, and this case is treated in a different way.

By using Delaunay's canonical elements a theorem is proved that there are no long-periodic terms of the first order in the expression of the semi-major axis.

1. *The disturbing function.* In the present paper it is assumed that air-drag is absent and that the gravitational field of the earth is axially symmetric. Under these assumptions the gravitational potential of the earth at a point where the geocentric distance and the latitude are r and δ, respectively, is expanded into the series of spherical harmonics,

$$U = \frac{GM}{r}\left\{1 + \frac{A_2}{r^2}\left(\frac{1}{3} - \sin^2\delta\right) + \frac{A_3}{r^3}\left(\frac{5}{2}\sin^2\delta - \frac{3}{2}\right)\sin\delta + \frac{A_4}{r^4}\left(\frac{3}{35} + \frac{1}{7}\sin^2\delta - \frac{1}{4}\sin^2 2\delta\right) + \cdots\right\}, \quad (1)$$

where G is the gravitational constant and M is the mass of the earth. The second and fourth terms in the expression of the potential are due to the oblateness of the earth, and the third term is due to the asymmetry with respect to the equatorial plane. A_2 is taken to be of the first order of small quantities, and A_3 and A_4 are to be of the second order. The coefficients of higher harmonics may be of the third order of small quantities or less. (O'Keefe, Eckels and Squires 1959, Kozai 1959).

The purpose of the present author is to derive the periodic perturbations of the first order and secular perturbations up to the second order. Therefore terms of higher order than the fifth in the potential series may be neglected.

As the satellite is always on an ellipse, whose position and shape are variable, it is convenient to express r and δ in (1) by elliptical elements of the satellite by the following relations:

$$r = \frac{a(1 - e^2)}{1 + e\cos v}, \qquad \sin\delta = \sin i \sin(v + \omega), \quad (2)$$

where a is the semi-major axis, e is the eccentricity, i is the inclination to the equator, ω is the argument of perigee and v is the true anomaly.

The disturbing function due to the oblateness of the earth is then

$$R = U - GM/r$$

$$= GM\left[\frac{A_2}{a^3}\left(\frac{a}{r}\right)^3\left\{\frac{1}{3} - \frac{1}{2}\sin^2 i + \frac{1}{2}\sin^2 i \cos 2(v+\omega)\right\}\right.$$

$$+ \frac{A_3}{a^4}\left(\frac{a}{r}\right)^4\left\{\left(\frac{15}{8}\sin^2 i - \frac{3}{2}\right)\sin(v+\omega) - \frac{5}{8}\sin^2 i \sin 3(v+\omega)\right\}\sin i$$

$$+ \frac{A_4}{a^5}\left(\frac{a}{r}\right)^5\left\{\frac{3}{35} - \frac{3}{7}\sin^2 i + \frac{3}{8}\sin^4 i + \sin^2 i\left(\frac{3}{7} - \frac{1}{2}\sin^2 i\right)\cos 2(v+\omega)\right.$$

$$\left.\left. + \frac{1}{8}\sin^4 i \cos 4(v+\omega)\right\}\right]. \quad (3)$$

The true anomaly, v, is easily transformed to the mean anomaly, M, which is a linear function of time in non-perturbed motion, by the differential equation

$$\frac{dv}{dM} = \frac{a^2}{r^2}\sqrt{1-e^2}; \quad (4)$$

r/a and v appearing in the disturbing function of (3) are then functions of e and M only, and are periodic with respect to M. Therefore, R is also a periodic function of M and ω. In R, terms depending neither on M nor on ω are called secular, terms depending on ω but not on M are long-periodic, and terms depending on M are short-periodic.

As the long-periodic perturbations originate from terms of the second order in R, we must retain secular terms and long-periodic terms up to the second order. However, for short-periodic terms we need only terms of the first order.

Using the following relations (Tisserand 1889):

$$\overline{\left(\frac{a}{r}\right)^3} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM = (1-e^2)^{-3/2},$$

$$\overline{\left(\frac{a}{r}\right)^3 \sin 2v} = \overline{\left(\frac{a}{r}\right)^3 \cos 2v} = 0,$$

$$\overline{\left(\frac{a}{r}\right)^4 \cos v} = e(1-e^2)^{-5/2},$$

$$\overline{\left(\frac{a}{r}\right)^4 \sin v} = \overline{\left(\frac{a}{r}\right)^4 \cos 3v} = \overline{\left(\frac{a}{r}\right)^4 \sin 3v} = 0, \quad (5)$$

$$\overline{\left(\frac{a}{r}\right)^5} = (1-e^2)^{-7/2}\left(1 + \frac{3}{2}e^2\right),$$

$$\overline{\left(\frac{a}{r}\right)^5 \cos 2v} = \frac{3}{4}e^2(1-e^2)^{-7/2},$$

$$\overline{\left(\frac{a}{r}\right)^5 \sin 2v} = \overline{\left(\frac{a}{r}\right)^5 \cos 4v} = \overline{\left(\frac{a^5}{r}\right) \sin 4v} = 0,$$

we can pick up only necessary terms by the above criterion as follows:

$$R_1 = GM\frac{A_2}{a^3}\left(\frac{1}{3}-\frac{1}{2}\sin^2 i\right)(1-e^2)^{-3/2},$$

$$R_2 = GM\frac{A_4}{a^5}\left(\frac{3}{35}-\frac{3}{7}\sin^2 i+\frac{3}{8}\sin^4 i\right)(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right),$$

$$\begin{aligned}R_3 = GM\Bigg\{&\frac{3}{2}\frac{A_3}{a^4}\sin i\left(\frac{5}{4}\sin^2 i-1\right)e(1-e^2)^{-5/2}\sin\omega \\ &+\frac{A_4}{a^5}\sin^2 i\left(\frac{9}{28}-\frac{3}{8}\sin^2 i\right)e^2(1-e^2)^{-7/2}\cos 2\omega\Bigg\},\end{aligned} \tag{6}$$

$$R_4 = GM\frac{A_2}{a^3}\left(\frac{a}{r}\right)^3\left[\left(\frac{1}{3}-\frac{1}{2}\sin^2 i\right)\left\{1-\left(\frac{r}{a}\right)^3(1-e^2)^{-3/2}\right\}+\frac{1}{2}\sin^2 i\cos 2(v+\omega)\right],$$

where R_1, R_2, R_3 and R_4 are first-order secular, second-order secular, long-periodic, and short-periodic parts of the disturbing function, respectively.

2. *Perturbations of short period.* The differential equations representing variations of orbital elements are:

$$\begin{aligned}\frac{da}{dt}&=\frac{2}{na}\frac{\partial R}{\partial M},\\ \frac{de}{dt}&=\frac{1-e^2}{na^2e}\frac{\partial R}{\partial M}-\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial\omega},\\ \frac{d\omega}{dt}&=-\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}+\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e},\\ \frac{di}{dt}&=\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial\omega},\\ \frac{d\Omega}{dt}&=\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i},\\ \frac{dM}{dt}&=n-\frac{1-e^2}{na^2e}\frac{\partial R}{\partial e}-\frac{2}{na}\frac{\partial R}{\partial a},\end{aligned} \tag{7}$$

where n is related to a by $n^2a^3 = GM$.

To derive short-periodic perturbations of the first order, after replacing R by R_4 in (7), one may regard a, n, e, i and ω on the right-hand sides of equations (7) to be constant. However, n, appearing as the first term of the last equation without any factor, is variable, but it is a known function of time, after obtaining the expression of the semi-major axis. True anomaly, v, may be regarded also to be a known function of time on the right-hand side, and one can transform the independent variable from t to v by

$$dt = \frac{dt}{dM}dM = \frac{1}{n}\left(\frac{r}{a}\right)^2\frac{1}{\sqrt{1-e^2}}dv. \tag{8}$$

Then, for example, the short-periodic perturbations of the inclination are obtained by,

$$di_s = \frac{\cos i}{n^2a^2(1-e^2)\sin i}\int\left(\frac{r}{a}\right)^2\frac{\partial R_4}{\partial\omega}dv, \tag{9}$$

where the integrand can be expressed by a finite trigonometric series and is, of course, integrable analytically within the necessary accuracy. The results for the six elements are as follows:

$$da_s = \frac{A_2}{a}\left[\frac{2}{3}\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(\frac{a}{r}\right)^3-(1-e^2)^{-3/2}\right\}+\left(\frac{a}{r}\right)^3\sin^2 i\cos 2(v+\omega)\right],$$

$$de_s = \frac{1-e^2}{e}\frac{A_2}{a^2}\left[\frac{1}{3}\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(\frac{a}{r}\right)^3-(1-e^2)^{-3/2}\right\}+\frac{1}{2}\left(\frac{a}{r}\right)^3\sin^2 i\cos 2(v+\omega)\right]$$
$$-\frac{\sin^2 i}{2e}\frac{A_2}{ap}\left\{\cos 2(v+\omega)+e\cos(v+2\omega)+\frac{1}{3}e\cos(3v+2\omega)\right\},$$

$$d\omega_s = \frac{A_2}{p^2}\left[\left(2-\frac{5}{2}\sin^2 i\right)(v-M+e\sin v)\right.$$
$$+\left(1-\frac{3}{2}\sin^2 i\right)\left\{\frac{1}{e}\left(1-\frac{1}{4}e^2\right)\sin v+\frac{1}{2}\sin 2v+\frac{e}{12}\sin 3v\right\}$$
$$-\frac{1}{e}\left\{\frac{1}{4}\sin^2 i+\left(\frac{1}{2}-\frac{15}{16}\sin^2 i\right)e^2\right\}\sin(v+2\omega)+\frac{e}{16}\sin^2 i\sin(v-2\omega)$$
$$-\frac{1}{2}\left(1-\frac{5}{2}\sin^2 i\right)\sin 2(v+\omega)+\frac{1}{e}\left\{\frac{7}{12}\sin^2 i-\frac{1}{6}\left(1-\frac{19}{8}\sin^2 i\right)e^2\right\}\sin(3v+2\omega) \quad (10)$$
$$\left.+\frac{3}{8}\sin^2 i\sin(4v+2\omega)+\frac{e}{16}\sin^2 i\sin(5v+2\omega)\right],$$

$$di_s = \frac{1}{4}\frac{A_2}{p^2}\sin 2i\left\{\cos 2(v+\omega)+e\cos(v+2\omega)+\frac{e}{3}\cos(3v+2\omega)\right\},$$

$$d\Omega_s = -\frac{A_2}{p^2}\cos i\left\{v-M+e\sin v-\frac{1}{2}\sin 2(v+\omega)-\frac{e}{2}\sin(v+2\omega)-\frac{e}{6}\sin(3v+2\omega)\right\},$$

$$e\,dM_s = \frac{A_2}{p^2}\sqrt{1-e^2}\left[-\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(1-\frac{e^2}{4}\right)\sin v+\frac{e}{2}\sin 2v+\frac{e^2}{12}\sin 3v\right\}\right.$$
$$+\sin^2 i\left\{\frac{1}{4}\left(1+\frac{5}{4}e^2\right)\sin(v+2\omega)-\frac{e^2}{16}\sin(v-2\omega)-\frac{7}{12}\left(1-\frac{e^2}{28}\right)\sin(3v+2\omega)\right.$$
$$\left.\left.-\frac{3}{8}e\sin(4v+2\omega)-\frac{e^2}{16}\sin(5v+2\omega)\right\}\right],$$

where

$$p = a(1-e^2).$$

As the mean values of $\cos jv$ ($j = 1, 2, \cdots$) with respect to M do not vanish, but

$$\overline{\cos jv} = \left(\frac{-e}{1+\sqrt{1-e^2}}\right)^j (1+j\sqrt{1-e^2}), \quad (11)$$

mean values of these short-periodic perturbations are not zero, except for those of a. Their mean values with respect to M are:

$$\overline{de_s} = \frac{A_2}{p^2}\sin^2 i\,\frac{1-e^2}{6e}\,\overline{\cos 2v}\,\cos 2\omega,$$

$$\overline{d\omega_s} = \frac{A_2}{p^2}\left\{\sin^2 i\left(\frac{1}{8}+\frac{1-e^2}{6e^2}\,\overline{\cos 2v}\right)+\frac{1}{6}\cos^2 i\,\overline{\cos 2v}\right\}\sin 2\omega,$$

$$\overline{di_s} = -\frac{1}{12}\frac{A_2}{p^2}\sin 2i\,\overline{\cos 2v}\,\cos 2\omega, \tag{12}$$

$$\overline{d\Omega_s} = -\frac{1}{6}\frac{A_2}{p^2}\cos i\,\overline{\cos 2v}\,\sin 2\omega,$$

$$\overline{dM_s} = -\frac{A_2}{p^2}\sqrt{1-e^2}\sin^2 i\left\{\frac{1}{8}+\frac{1+\frac{e^2}{2}}{6e^2}\,\overline{\cos 2v}\right\}\sin 2\omega.$$

Therefore, $de_s - \overline{de_s}$, $d\omega_s - \overline{d\omega_s}$, $di_s - \overline{di_s}$, $dM_s - \overline{dM_s}$ and $d\Omega_s - \overline{d\Omega_s}$ must be the short-periodic perturbations whose mean values with respect to the mean anomaly are zero.

Expressions of the mean anomaly and the argument of perigee are rather complicated, so it is better to combine the four elements a, e, ω and M into the radius vector r and the argument of latitude $L = v + \omega$ by the following relations:

$$\frac{dr}{a} = \frac{e}{\sqrt{1-e^2}}\sin v\,dM + \frac{r}{a}\frac{da}{a} - \cos v\,de,$$

$$dv = \frac{a^2}{r^2}\sqrt{1-e^2}\,dM + \sin v\left(1+\frac{r}{p}\right)\frac{a}{r}\,de.$$

Putting

$$da = -a_0\frac{A_2}{p^2}\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2} + da_s, \qquad de = de_s,$$

$$d\omega = d\omega_s - \frac{3}{8}\frac{A_2}{p^2}\sin^2 i\sin 2\omega,$$

$$dM = dM_s + \frac{3}{8}\frac{A_2}{p^2}\sqrt{1-e^2}\sin^2 i\sin 2\omega,$$

the deviations of the radius vector and argument of latitude of the satellite, from those computed by mean orbital elements are obtained as follows:

$$\frac{dr}{a} = \frac{1}{3}\frac{A_2}{ap}\left(1-\frac{3}{2}\sin^2 i\right)\left\{-1-\frac{1}{e}\left(1-\sqrt{1-e^2}\right)\cos v+\frac{r}{a}\frac{1}{\sqrt{1-e^2}}\right\}+\frac{1}{6}\frac{A_2}{ap}\sin^2 i\cos 2(v+\omega),$$

$$\begin{aligned} dL = \frac{A_2}{p^2}\Bigg[&\left(2-\frac{5}{2}\sin^2 i\right)(v-M+e\sin v)\\ &+\left(1-\frac{3}{2}\sin^2 i\right)\left\{\frac{2}{3e}\left(1-\frac{e^2}{2}-\sqrt{1-e^2}\right)\sin v+\frac{1}{6}\left(1-\sqrt{1-e^2}\right)\sin 2v\right\}\\ &-\left(\frac{1}{2}-\frac{5}{6}\sin^2 i\right)e\sin(v+2\omega)-\left(\frac{1}{2}-\frac{7}{12}\sin^2 i\right)\sin 2(v+\omega)-\frac{e}{6}\cos^2 i\sin(3v+2\omega)\Bigg]. \end{aligned} \tag{13}$$

The secular perturbations of the first order are easily derived by putting $R = R_1$ in (7), as

$$\bar{\omega} = \omega_0 + \frac{A_2}{p^2}\bar{n}\left(2 - \frac{5}{2}\sin^2 i\right)t,$$

$$\bar{\Omega} = \Omega_0 - \frac{A_2}{p^2}\bar{n}t\cos i$$

$$\bar{M} = M_0 + \bar{n}t,$$

$$\bar{n} = n_0 + \frac{A_2}{p^2}n_0\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1 - e^2},$$

where ω_0, Ω_0 and M_0 are mean values at the epoch, that is, the initial values, from which periodic perturbations have been subtracted. n_0 is the unperturbed mean motion, which is related to the unperturbed semi-major axis a_0 by $n_0^2 a_0^3 = GM$.

It is more convenient to adopt as a mean value of the semi-major axis not a_0 but

$$\bar{a} = a_0\left\{1 - \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1 - e^2}\right\},$$

so that the following relation holds:

$$\bar{n}^2\bar{a}^3 = GM\left\{1 - \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1 - e^2}\right\}. \tag{14}$$

And to derive the expressions of (13) this value of $\bar{a}$ has been already adopted as a mean semi-major axis.

3. *A theorem on non-existence of long-periodic terms in the semi-major axis.* As is well known, Poisson proved a theorem of non-existence of secular perturbations of the semi-major axis of a planetary orbit. However, there are some differences between theories of planetary and satellite motions. Because the perihelion and node of an ordinary planet moves around the sun with the period of some ten thousand years, one may expand $e\sin\omega$, $e\cos\omega$, $i\sin\Omega$ and $i\cos\Omega$ into power series of time in the planetary theory, so that terms depending only on ω and Ω must be regarded to be secular, not long-periodic. From this point of view Poisson proved the theorem. Now we observe the motion of an earth satellite for a long interval of time, in which the line of apside makes several revolutions. So the corresponding theorem in the satellite motion is that there are no long-periodic terms in the expression of the semi-major axis.

Let us transform variables from Kepler's elements to Delaunay's canonical ones:

$$\begin{cases} L = \sqrt{\mu a}, & G = \sqrt{\mu a(1 - e^2)}, & H = \sqrt{\mu a(1 - e^2)}\cos i, \\ l = M, & g = \omega, & h = \Omega, \end{cases} \tag{15}$$

where $\mu = GM$.

These variables must satisfy the following canonical equations:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dH}{dt} &= \frac{\partial F}{\partial h}, \\ \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}, \end{aligned} \tag{16}$$

where

$$F = \frac{\mu^2}{2L^2} + R.$$

Since F does not depend on h, there is an integral H = constant. Variations of the semi-major axis are represented by:

$$\frac{dL}{dt} = \frac{\partial F}{\partial l} = \frac{\partial R_4}{\partial l},$$

where $\mu^2/2L^2$, R_1, R_2 and R_3 are omitted in F because they do not depend on l.

Long-periodic perturbations of the first order of L will come from long-periodic terms of the second order of $\partial R_4/\partial l$, which may be expanded into Taylor's series,

$$\frac{\partial R_4}{\partial l} = \left(\frac{\partial R_4}{\partial l}\right) + \left(\frac{\partial^2 R_4}{\partial l \partial L}\right) dL + \left(\frac{\partial^2 R_4}{\partial l^2}\right) dl + \left(\frac{\partial^2 R_4}{\partial l \partial G}\right) dG + \left(\frac{\partial^2 R_4}{\partial l \partial g}\right) dg + \cdots \qquad (17)$$

where terms of higher than the third order are neglected and in parentheses l, L, g and G are replaced by their respective mean values, $l = \bar{n}t + l_0$, $L = \bar{L}_0$, $g = \dot{g}t + g_0$ and $G = \bar{G}$; dl, dL, dG and dg are deviations of instantaneous values from their respective mean values and are of the first order. $\bar{n}$ and $\dot{g}$ are secular motions of the mean longitude and the longitude of perigee, and are supposed to be known.

In equation (17) one must consider only long-periodic terms of the second order, and after integrating with respect to time only such terms of the first order, omiting other terms. The first term of (17), $\partial R_4/\partial l$, cannot have any long-periodic terms of any order, so must be omitted. As R_4 depends on t only through M within the accuracy of the first order, the following relations hold:

$$\int\left(\frac{\partial^2 R_4}{\partial l^2}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial l}\right), \qquad \int\left(\frac{\partial^2 R_4}{\partial l \partial L}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial L}\right),$$
$$\int\left(\frac{\partial^2 R_4}{\partial l \partial G}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial G}\right), \qquad \int\left(\frac{\partial^2 R_4}{\partial l \partial g}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial g}\right). \qquad (18)$$

Integrating $\partial R_4/\partial l$ in (17) with respect to time by parts using the relations (18), one has

$$\begin{aligned} dL &= \int \frac{\partial R_4}{\partial l}\, dt \\ &= \frac{1}{\bar{n}}\left[\left(\frac{\partial R_4}{\partial L}\right) dL + \left(\frac{\partial R_4}{\partial l}\right) dl + \left(\frac{\partial R_4}{\partial G}\right) dG + \left(\frac{\partial R_4}{\partial g}\right) dg\right] \\ &\quad - \frac{1}{\bar{n}}\int\left\{\left(\frac{\partial R_4}{\partial L}\right)\frac{dL}{dt} + \left(\frac{\partial R_4}{\partial l}\right)\left(\frac{dl}{dt} - \bar{n}\right) + \left(\frac{\partial R_4}{\partial G}\right)\frac{dG}{dt} + \left(\frac{\partial R_4}{\partial g}\right)\left(\frac{dg}{dt} - \dot{g}\right)\right\} dt. \qquad (19) \end{aligned}$$

After integration the first part contains only terms of the second order, so can be omitted. The second part is transformed to

$$\begin{aligned} dL &= -\frac{1}{\bar{n}}\int\left[\left(\frac{\partial R_4}{\partial L}\right)\left(\frac{\partial R_4}{\partial l}\right) + \left(\frac{\partial R_4}{\partial l}\right)\left\{dn - \left(\frac{\partial R_4}{\partial L}\right)\right\}\right. \\ &\qquad \left. + \left(\frac{\partial R_4}{\partial G}\right)\left(\frac{\partial R_4}{\partial g}\right) + \left(\frac{\partial R_4}{\partial g}\right)\left(\frac{\partial R_4}{\partial G}\right)\right] dt \\ &= -\frac{1}{\bar{n}}\int dn\left(\frac{\partial R_4}{\partial l}\right) dt. \qquad (20) \end{aligned}$$

By using the relation $nL^3 = \text{const.}$, this equation is written as

$$dL = \frac{3}{L}\int\left(\frac{\partial R_4}{\partial l}\right)dLdt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{\bar{n}L}\int(R_4)\frac{dL}{dt}dt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{\bar{n}L}\int(R_4)\left(\frac{\partial R_4}{\partial l}\right)dt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{2\bar{n}L}\int\left(\frac{\partial R_4^2}{\partial l}\right)dt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{2\bar{n}^2L}(R_4^2). \tag{21}$$

Since the terms are all of the second order after integration, one can conclude that there are no long-periodic perturbations of the first order in the expression of the semi-major axis. Of course there are no secular terms in the semi-major axis.

4. *Secular perturbations of the second order and long-periodic perturbations.* Let E_i be one of the six orbital elements of a satellite, and express its variation by the differential equation

$$\frac{dE_i}{dt} = f_i. \tag{22}$$

A function f_i may be expanded into a power series of deviations from the mean orbital elements as

$$\frac{dE_i}{dt} = (f_i) + \sum_j\left(\frac{\partial f_i}{\partial E_j}\right)dE_j + \cdots; \tag{23}$$

Secular perturbations of the second order and long-periodic perturbations of the first order will come from the terms of the second order on the right-hand side of (23). However, the complete first order expression of dE_j is not yet known except for da. By integrating equation (23) by parts as in the previous section, there holds,

$$dE_i = \int(f_i)dt + \sum_j[F_{ij}dE_j] - \sum_j\int F_{ij}\frac{dE_j}{dt}dt, \tag{24}$$

where

$$F_{ij} = \int\left(\frac{\partial f_i}{\partial E_j}\right)dt.$$

If E_i is the inclination, i, for example,

$$f_i = \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial\omega}.$$

In this case $\partial f_i/\partial E_j$ has neither long-periodic nor secular terms as far as the terms of the first order are concerned. As it is easily proved that i and e cannot contain any secular terms, $\sum_j F_{ij}dE_j$ can be omitted from the equation of the inclination, because this is of the second order and has no secular terms.

Therefore all terms on the right-hand side of (23) for the inclination are known and long-periodic terms can be picked up by using the following relations (Tisserand 1889):

$$\overline{\left(\frac{a}{r}\right)^6} = (1 - e^2)^{-9/2}\left(1 + 3e^2 + \frac{3}{8}e^4\right),$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos v} = 2e(1 - e^2)^{-9/2}\left(1 + \frac{3}{4}e^2\right),$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos 2v} = \frac{3}{2}e^2(1 - e^2)^{-9/2}\left(1 + \frac{1}{6}e^2\right),$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos 3v} = \frac{e^3}{2}(1 - e^2)^{-9/2},$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos 4v} = \frac{e^4}{16}(1 - e^2)^{-9/2},$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos jv} = 0, \quad (j > 4),$$

$$\overline{\left(\frac{a}{r}\right)^3} = (1 - e^2)^{-3/2},$$

$$\overline{\left(\frac{a}{r}\right)^3 \cos v} = \frac{e}{2}(1 - e^2)^{-3/2},$$

$$\overline{\left(\frac{a}{r}\right)^3 \cos jv} = 0, \quad (j > 1).$$

Then the expression of long-periodic perturbations of the inclination are derived as:

$$di_1 = \overline{di_s} - \frac{A_2}{p^2}\frac{e^2 \sin 2i}{8(4-5\sin^2 i)}\left\{\frac{14-15\sin^2 i}{6} - \frac{A_4}{A_2{}^2}\frac{18-21\sin^2 i}{7}\right\}\cos 2\omega - \frac{3}{4}\frac{A_3}{A_2 p} e \cos i \sin \omega. \quad (25)$$

Now there is an integral, $\sqrt{a(1 - e^2)} \cos i$ = const., and it has been already proved that there are no long-periodic perturbations of the first order in the semi-major axis, so de_1, the long-periodic perturbation of the eccentricity, is

$$de_1 = -\frac{1 - e^2}{e}\frac{\sin i}{\cos i} di_1$$

$$= \overline{de_s} + \frac{A_2}{pa}\frac{e \sin^2 i}{4(4-5\sin^2 i)}\left\{\frac{14-15\sin^2 i}{6} - \frac{A_4}{A_2{}^2}\frac{18-21\sin^2 i}{7}\right\}\cos 2\omega + \frac{3}{4}\frac{A_3}{A_2 a}\sin i \sin \omega. \quad (26)$$

However, the same principle cannot be applied to obtain the expressions of the node and the argument of perigee, because in these cases $\partial f_i/\partial i$, $\partial f_i/\partial e$ and $\partial f_i/\partial a$ have secular terms. But fortunately complete expressions of di, de and da have already been derived. And since it is also proved that $F_{j\omega}d\omega$ and $F_{jM}dM$ do not include secular terms, the following expressions are derived:

$$\dot{\Omega} = -\frac{A_2}{\bar{p}^2}\bar{n}\cos\bar{i}\left[1 + \frac{A_2}{p^2}\left\{\frac{3}{2} + \frac{e^2}{6} - 2\sqrt{1 - e^2} - \sin^2 i\left(\frac{5}{3} - \frac{5}{24}e^2 - 3\sqrt{1 - e^2}\right)\right\}\right]$$
$$- \frac{A_4}{p^4} n \cos i \frac{12-21\sin^2 i}{14}\left(1 + \frac{3}{2}e^2\right), \quad (27)$$

$$\dot{\omega} = \frac{A_2}{\bar{p}^2}\bar{n}\left(2-\frac{5}{2}\sin^2\bar{i}\right)\left[1+\frac{A_2}{p^2}\left\{2+\frac{e^2}{2}-2\sqrt{1-e^2} - \sin^2 i\left(\frac{43}{24}-\frac{e^2}{48}-3\sqrt{1-e^2}\right)\right\}\right] - \frac{5}{12}\frac{A_2^2}{p^4}e^2 n\cos^4 i$$
$$+\frac{A_4}{p^4}n\left[\frac{12}{7}-\frac{93}{14}\sin^2 i+\frac{21}{4}\sin^4 i+e^2\left(\frac{27}{14}-\frac{189}{28}\sin^2 i+\frac{81}{16}\sin^4 i\right)\right],$$

where $\bar{i}$ and $\bar{e}$ are mean values of the inclination and the eccentricity over all the periods, and

$$\bar{p} = \bar{a}(1-\bar{e}^2).$$

$$d\Omega_1 = \overline{d\Omega_s} - \frac{A_2}{p^2}\frac{e^2\cos i}{2(4-5\sin^2 i)}\left[\left\{\frac{7-15\sin^2 i}{6}-\frac{A_4}{A_2^2}\frac{9-21\sin^2 i}{7}\right\}\right.$$
$$\left.+\frac{5\sin^2 i}{2(4-5\sin^2 i)}\left\{\frac{14-15\sin^2 i}{6}-\frac{A_4}{A_2^2}\frac{18-21\sin^2 i}{7}\right\}\right]\sin 2\omega+\frac{3}{4}\frac{A_3}{A_2 p}\frac{\cos i}{\sin i}e\cos\omega,$$

$$d\omega_1 = \overline{d\omega_s} - \frac{3}{8}\frac{A_2}{p^2}\sin^2 i\sin 2\omega$$
$$-\frac{A_2}{p^2}\left[\frac{1}{4-5\sin^2 i}\left\{\frac{14-15\sin^2 i}{24}\sin^2 i - e^2\frac{28-158\sin^2 i+135\sin^4 i}{48}\right.\right.$$
$$\left.-\frac{A_4}{A_2^2}\left(\frac{18-21\sin^2 i}{28}\sin^2 i - e^2\frac{36-210\sin^2 i+189\sin^4 i}{56}\right)\right\}$$
$$\left.-\frac{e^2\sin^2 i(13-15\sin^2 i)}{(4-5\sin^2 i)^2}\left(\frac{14-15\sin^2 i}{24}-\frac{A_4}{A_2^2}\frac{18-21\sin^2 i}{28}\right)\right]\sin 2\omega$$
$$+\frac{3}{4}\frac{A_3}{A_2 p}\frac{\sin^2 i - e^2\cos^2 i}{\sin i}\frac{1}{e}\cos\omega.$$

It is very difficult to derive the long-periodic perturbations of the first order of the mean anor because the long-periodic perturbations of the second order of the semi-major axis are not kn However, usually the atmosphere of the earth changes the mean motion of the actual satelli much that any long-periodic variations of the mean anomaly cannot be detected from observa with good accuracy.

The results are then:

$$a = \bar{a} + da_s, \qquad \bar{a} = a_0\left\{1-\frac{A_2}{p^2}\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right\},$$

$$e = \bar{e} + de_s - \overline{de_s} + de_1,$$

$$i = \bar{i} + di_s - \overline{di_s} + di_1,$$

$$\omega = \omega_0 + \dot{\omega}t + d\omega_s - \overline{d\omega_s} + d\omega_1,$$

$$\Omega = \Omega_0 + \dot{\Omega}t + d\Omega_s - \overline{d\Omega_s} + d\Omega_1,$$

$$M = M_0 + \bar{n}t + dM_s, \qquad \bar{n} = n_0\left\{1+\frac{A_2}{p^2}\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right\}.$$

$$n_0^2 a_0^3 = GM,$$

e $\bar{e}$ and $\bar{\imath}$ are mean values with respect to M , and ω_0, Ω_0 and M_0 are initial values from n periodic perturbations have been sub- ed. On the right-hand sides of these expres- one must always use constant mean values e, i, $\omega - \dot{\omega}t$, $\Omega - \dot{\Omega}t$ and $M - \bar{n}t$.

The case when the inclination or eccentricity y small. In the previous sections it is as- d that $4-5 \sin^2 i$, i and e are not very small. s very small, periodic perturbations of M become very large. However, in the case e short-periodic perturbations the difficulty vanish, if elements are transformed from M to r and $v + \omega$.

r the long-periodic perturbations the vari- must be transformed from e and ω to

$$\xi = e \cos \omega,$$

$$\eta = -e \sin \omega.$$

e variables must satisfy

$$= \frac{\partial R}{\partial \eta} = \dot{\omega}\eta - \frac{3}{2}\frac{A_3}{a^3}\sin i\left(\frac{5}{4}\sin^2 i - 1\right),$$

$$= -\frac{\partial R}{\partial \xi} = -\dot{\omega}\xi, \tag{32}$$

e $\dot{\omega}$ has the same value as in the previous on.

e solutions of these equations are

$$e \cos \omega = e_0 \cos \bar{\omega},$$

$$e \sin \omega = e_0 \sin \bar{\omega} + \frac{3}{4}\frac{A_3}{aA_2}\sin i,$$

e

$$\bar{\omega} = \dot{\omega}t + \omega_0.$$

d ω_0 are constants of integration.

e_0 is much smaller than $3A_3/4aA_2$, which is e first order, the argument of perigee cannot e around the earth completely, but oscillates nd the value 90° as follows:

$$\omega = 90^\circ + \frac{4}{3}\frac{aA_2e_0}{A_3 \sin i}\cos \bar{\omega}.$$

The variation of the eccentricity is expressed by

$$e = \frac{3}{4}\frac{A_3}{aA_2}\sin i + e_0 \sin \bar{\omega}.$$

In this case $d\Omega_1 = di_1 = 0$, because they have a factor e.

When the inclination is very small, instead of ω, the longitude of perigee, $\pi = \omega + \Omega$ is adopted. The longitude of perigee moves secularly as

$$\pi = \dot{\pi}t + \pi_0,$$

where

$$\dot{\pi} = \frac{A_2}{pa^2}\bar{n}.$$

The variations of $i \sin \Omega$ and $i \cos \Omega$ are derived as in the previous case as

$$i \sin \Omega = i_0 \sin \bar{\Omega} + \frac{3}{4}\frac{A_3}{A_2 p} e \cos \pi,$$

$$i \cos \Omega = i_0 \cos \bar{\Omega} + \frac{3}{4}\frac{A_3}{A_2 p} e \sin \pi,$$

where

$$\bar{\Omega} = \dot{\Omega}t + \Omega_0.$$

If i_0 is much smaller than $3eA_3/4A_2p$, then

$$\Omega = \pi + 90^\circ + \frac{4}{3}\frac{A_2 p i_0}{A_3 e}\cos(\pi - \bar{\Omega}),$$

$$i = \frac{3}{4}\frac{A_3}{A_2 p} e + i_0 \cos(\pi - \bar{\Omega}).$$

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SOLUTION OF THE PROBLEM OF ARTIFICIAL SATELLITE THEORY WITHOUT DRAG

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Abstract. Sections 1–6 give the solution of the main problem for a spheroidal earth with the potential limited to the principal term and the second harmonic which contains the small factor k_2. The solution is developed in powers of k_2 in canonical variables by a method which is basically the same as that used in treating a different problem by von Zeipel (1916). The periodic terms are divided in two classes: the short-period terms contain the mean anomaly in their arguments; the arguments of the long-period terms are multiples of the mean argument of the perigee.

The periodic terms, both of short and long period, are developed to $O(k_2)$; the secular motions are obtained to $O(k_2^2)$. The results are obtained in closed form; no series developments in eccentricity or inclination arise. The solution does not apply to orbits near the critical inclination, 63°.4, but is otherwise valid for any eccentricity < 1 and any inclination.

Section 7 gives the long-period terms and the additions to the secular motions caused by the fourth harmonic in the potential; section 8 gives the contributions by the third and fifth harmonics; section 9 contains formulas for computation.

1. *The equations of motion.* The equations of motion of a small mass attracted by a spheroid are

$$\frac{d^2x}{dt^2}=\frac{\partial U}{\partial x},\quad \frac{d^2y}{dt^2}=\frac{\partial U}{\partial y},\quad \frac{d^2z}{dt^2}=\frac{\partial U}{\partial z},$$

with

$$U=\frac{\mu}{r}+\frac{\mu k_2}{r^3}(1-3\sin^2\beta)+\frac{\mu k_4}{r^5}\left(1-10\sin^2\beta+\frac{35}{3}\sin^4\beta\right)+\cdots.$$

The equatorial plane of the spheroid is taken as the xy plane; β is the latitude, and if M is the mass of the spheroid and k the Gaussian constant, $\mu = k^2M$.

The main problem of artificial satellite theory may be considered the problem with $k_4 = 0$. Let I be the instantaneous inclination of the orbital plane with the equatorial plane, g the distance of the pericenter from the ascending node, and f the true anomaly. Then

$$\sin\beta=\sin I\sin(g+f),$$
$$2\sin^2\beta=\sin^2 I\,[1-\cos(2g+2f)],$$

and the disturbing function can be written

$$R=\frac{\mu k_2}{a^3}\left[\left(-\tfrac{1}{2}+\tfrac{3}{2}\cos^2 I\right)\frac{a^3}{r^3}+\left(\tfrac{3}{2}-\tfrac{3}{2}\cos^2 I\right)\frac{a^3}{r^3}\cos(2g+2f)\right].$$

Let a, e be the osculating semi-major axis and eccentricity, respectively. The Delaunay variables are then

$L=(\mu a)^{\frac{1}{2}}$, l=mean anomaly,

$G=L(1-e^2)^{\frac{1}{2}}$, g=argument of the pericenter,

$H=G\cos I$, h=longitude of ascending node.

With these variables the equations become

$$\frac{dL}{dt}=\frac{\partial F}{\partial l},\quad \frac{dl}{dt}=-\frac{\partial F}{\partial L},$$
$$\frac{dG}{dt}=\frac{\partial F}{\partial g},\quad \frac{dg}{dt}=-\frac{\partial F}{\partial G},\tag{1}$$
$$\frac{dH}{dt}=\frac{\partial F}{\partial h},\quad \frac{dh}{dt}=-\frac{\partial F}{\partial H}$$

$$F=\frac{\mu^2}{2L^2}+\frac{\mu^4k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}+\left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}\cos(2g+2f)\right].\tag{2}$$

The parts of F not exhibited in terms of the Delaunay variables may be expanded in Fourier series as follows:

$$\frac{a^3}{r^3}=\frac{L^3}{G^3}+\sum_{j=1}^{\infty}2P_j\cos jl\equiv\frac{L^3}{G^3}+\sigma_1,$$

$$\frac{a^3}{r^3}\cos(2g+2f)=\sum_{j=-\infty}^{+\infty}Q_j\cos(2g+jl)\equiv\sigma_2.$$

The coefficients P_j, Q_j are power series in the eccentricity e; the lowest power of e is e^j in P_j; $e^{|j-2|}$ in Q_j, $j\neq 0$. Since $e^2=1-G^2/L^2$, the derivatives of a function $\psi(e)$ with respect to L or G may be obtained by

$$\frac{\partial\psi}{\partial L} = \frac{1}{e}\frac{\partial\psi}{\partial e}\frac{G^2}{L^3},$$
$$\frac{\partial\psi}{\partial G} = -\frac{1}{e}\frac{\partial\psi}{\partial e}\frac{G}{L^2} \tag{3}$$

The functions ψ that will arise are functions of a/r and f. Well-known formulas of elliptic motion are

$$\frac{\partial}{\partial e}\left(\frac{a}{r}\right) = \frac{a^2}{r^2}\cos f, \tag{4}$$

$$\frac{\partial f}{\partial e} = \left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f. \tag{5}$$

Some important properties of the Hamiltonian F may be noted:

(a) the independent variable t is not explicitly present in F, hence the integral $F =$ constant exists;
(b) the variable h is not present in F;
(c) the coefficient Q_0 in σ_2 is zero.

The value of this coefficient is the constant term of the expansion in terms of the mean anomaly of $a^3r^{-3}\cos 2f$: hence

$$Q_0 = \frac{1}{\pi}\int_0^\pi \frac{a^3}{r^3}\cos 2f dl.$$

By making use of the integral of areas in the form

$$dl = \frac{L}{G}\frac{r^2}{a^2}df$$

this may be written

$$\begin{aligned} Q_0 &= \frac{1}{\pi}\frac{L}{G}\int_0^\pi \frac{a}{r}\cos 2f df \\ &= \frac{1}{\pi}\frac{L^3}{G^3}\int_0^\pi (1 + e\cos f)\cos 2f df \\ &= 0. \end{aligned}$$

This derivation may be generalized to show that the constant term of $(a^p/r^p)\cos qf$ (p and q positive integers), expanded in terms of the mean anomaly, is zero if $q > p - 2 \geqq 0$.

2. *Outline of the method of solution.* Consider a transformation from the variables L, G, H, l, g, h to new variables L', G', H', l', g', h', with the aid of a determining function $S(L', G', H', l, g, h)$. Then, if

$$L = \frac{\partial S}{\partial l}, \quad G = \frac{\partial S}{\partial g}, \quad H = \frac{\partial S}{\partial h},$$
$$l' = \frac{\partial S}{\partial L'}, \quad g' = \frac{\partial S}{\partial G'}, \quad h' = \frac{\partial S}{\partial H'},$$

the equations in the new variables will be

$$\begin{aligned} \frac{dL'}{dt} &= \frac{\partial F^*}{\partial l'}, & \frac{dl'}{dt} &= -\frac{\partial F^*}{\partial L'}, \\ \frac{dG'}{dt} &= \frac{\partial F^*}{\partial g'}, & \frac{dg'}{dt} &= -\frac{\partial F^*}{\partial G'}, \\ \frac{dH'}{dt} &= \frac{\partial F^*}{\partial h'}, & \frac{dh'}{dt} &= -\frac{\partial F^*}{\partial H'}, \end{aligned} \tag{7}$$

with

$$F^*(L', G', H', l', g', —) = F(L, G, H, l, g, —) \tag{8}$$

The dashes in the places for h' and h are used to indicate the absence of these variables.

The problem would be completely solved if a determining function were found such that F^* is a function of the variables L', G', H' only. The differential equations (7) show that then L', G', H' are constants, while l', g', h' are linear functions of the time. Substitution of this solution of the primed quantities into (6) then yields expressions from which the original variables may be obtained in terms of L', G', H', l', g', h' and therefore in terms of t and the constants of integration.

In the present problem it is more convenient to choose the determining function S in such a manner that l' is not present in F^*, while g' is permitted to appear. If this is accomplished, L' and H' will be constants, and the system is essentially reduced to one of one degree of freedom:

$$\frac{dG'}{dt} = \frac{\partial F^*}{\partial g'}, \quad \frac{dg'}{dt} = -\frac{\partial F^*}{\partial G'}$$

in which L' and H' are present as constants. After this system is solved, l', h' are obtained by quadratures from

$$\frac{dl'}{dt} = -\frac{\partial F^*}{\partial L'}, \quad \frac{dh'}{dt} = -\frac{\partial F^*}{\partial H'}.$$

Preferably a canonical transformation by the choice of a suitable determining function leads to an equivalent result.

3. *First-order solution.* The determining function is obtained by a method used by H. v. Zeipel (1916) in a qualitative study of the motions of minor planets. In this procedure a development in powers of the small parameter k_2 is introduced. If F_1 is written for R, the original Hamiltonian is

$$F = F_0 + F_1,$$

in which the subscript denotes the power of k_2 present as a factor. F_0 is a function of L only. Let also

$$S = S_0 + S_1 + S_2 \cdots,$$
$$F^* = F_0^* + F_1^* + F_2^* \cdots.$$

It will be convenient to choose

$$S_0 = L'l + G'g + H'h.$$

If then the expressions for L, G, H, l', g', h' given by (6) are substituted into

$$F(L, G, H, l, g, —) = F^*(L', G', H', —, g', —),$$

there results

$$F_0\left(\frac{\partial S}{\partial l}\right) + F_1\left(\frac{\partial S}{\partial l}, \frac{\partial S}{\partial g}, \frac{\partial S}{\partial h}, l, g, —\right)$$
$$= F_0^* + F_1^*\left(L', G', H', —, \frac{\partial S}{\partial G'}, —\right) + F_2^*\left(L', G', H', —, \frac{\partial S}{\partial G'}, —\right).$$

Expanding everywhere by Taylor's theorem to the second power of k_2, the result is

$$F_0(L') + \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + F_1(L', G', H', l, g, —) + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g}$$
$$= F_0^* + F_1^*(L', G', H', —, g, —) + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} + F_2^*(L', G', H', —, g, —).$$

Parts of corresponding order in k_2 on both sides yield:

order 0,

$$F_0(L') = F_0^*(L'); \tag{9}$$

order 1,

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + F_1 = F_1^*; \tag{10}$$

order 2,

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} = F_2^* + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'}. \tag{11}$$

The expansion could be carried on indefinitely, but for current practical requirements consideration of parts beyond the second order in k_2 appears to be unnecessary. Put

$$F_1 = F_{1s} + F_{1p},$$

in which F_{1s} is the part independent of l, F_{1p} the part dependent on l. In the terminology of planetary theory F_{1s} would be the secular part of the disturbing function, F_{1p} the periodic part. Hence, if for the sake of brevity the notation

$$A = -\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}, \quad B = +\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2} \tag{12}$$

is used,

$$F_{1s} = \frac{\mu^4 k_2}{L^3 G^3} A,$$

$$F_{1p} = \frac{\mu^4 k_2}{L^6}(A\sigma_1 + B\sigma_2).$$

Now equation (10) is split up into two equations

$$\frac{\partial S_1}{\partial l} = \frac{\mu^2 k_2}{L'^3}(A\sigma_1 + B\sigma_2),$$
$$F_1^* = \frac{\mu^4 k_2}{L'^3 G'^3} A, \tag{13}$$

in which use is made of

$$\frac{\partial F_0}{\partial L'} = -\frac{\mu^2}{L'^3}.$$

By integration S_1 may be obtained in the form of an infinite series as

$$S_1 = \frac{\mu^2 k_2}{L'^3}\left[A\sum_1^\infty \frac{2}{j} P_j \sin jl + B\sum_{-\infty}^{+\infty}\frac{1}{j} Q_j \sin(2g + jl)\right]. \tag{14}$$

No constants of integration are required, since only partial derivatives of S_1 with respect to L', G', H', l, g, h will be needed.

For the following developments it will be advantageous to obtain a closed expression for S_1. The method used at the end of Section 2 for obtaining the integral of functions of the type $a^p r^{-p} \cos qf$ can be used to obtain

$$\int \sigma_1 dl = \int \left(\frac{a^3}{r^3} - \frac{L^3}{G^3}\right) dl$$
$$= \frac{L^3}{G^3}[f - l + e \sin f],$$

$$\int \sigma_2 dl = \int \frac{a^3}{r^3} \cos (2g + 2f) dl$$
$$= \frac{L^3}{G^3}\left[\frac{1}{2} \sin (2g + 2f) + \frac{e}{2} \sin (2g + f) + \frac{e}{6} \sin (2g + 3f)\right].$$

Hence

$$S_1 = \frac{\mu^2 k_2}{G'^3}\left\{A(f - l + e \sin f) + B\left[\frac{1}{2} \sin (2g + 2f) + \frac{e}{2} \sin (2g + f) + \frac{e}{6} \sin (2g + 3f)\right]\right\}, \quad (15)$$

$$\frac{\partial S_1}{\partial g} = \frac{\mu^2 k_2}{G'^3} B\left[\cos (2g + 2f) + e \cos (2g + f) + \frac{e}{3} \cos (2g + 3f)\right]. \quad (16)$$

Introduce

$$\gamma_2 = \frac{\mu^2 k_2}{L'^4};$$

then

$$L = L' + \frac{\partial S_1}{\partial l}$$
$$= L'\left\{1 + \gamma_2\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^3}{r^3} - \frac{L'^3}{G'^3}\right) + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G'^2}\right)\frac{a^3}{r^3} \cos (2g + 2f)\right]\right\}, \quad (17)$$

$$G = G' + \frac{\partial S_1}{\partial g},$$
$$= G'\left\{1 + \gamma_2 \frac{L'^4}{G'^4}\left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G'^2}\right)\left[\cos (2g + 2f) + e \cos (2g + f) + \frac{e}{3} \cos (2g + 3f)\right]\right\} \quad (18)$$

$$H = H'. \quad (19)$$

S_1 was obtained as a function of G', H, e, f, g. Hence L' is present through e and f only. To obtain $\partial S_1/\partial L'$ it is convenient to obtain first

$$\frac{\partial S_1}{\partial e} = \frac{\mu^2 k_2}{G'^3}\left\{A\left[(1 + e \cos f)\frac{\partial f}{\partial e} + \sin f\right] + B\left[\cos (2g + 2f)(1 + e \cos f)\frac{\partial f}{\partial e} + \frac{1}{2} \sin (2g + f) + \frac{1}{6} \sin (2g + 3f)\right]\right\},$$

in which use was made of

$$\cos (2g + 2f) + \frac{e}{2} \cos (2g + f) + \frac{e}{2} \cos (2g + 3f) = \cos (2g + 2f)(1 + e \cos f).$$

With

$$(1 + e \cos f)\frac{\partial f}{\partial e} = \left(\frac{a^2}{r^2}\frac{G^2}{L^2} + \frac{a}{r}\right) \sin f$$

there results

$$\frac{\partial S_1}{\partial e} = \frac{\mu^2 k_2}{G'^3}\left\{A\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r} + 1\right) \sin f + \tfrac{1}{2}B\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2} - \frac{a}{r} + 1\right) \sin (2g + f) + \left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r} + \frac{1}{3}\right) \sin (2g + 3f)\right]\right\}.$$

With the aid of the relations (3) there follows at once

$$l = l' - \frac{\partial S_1}{\partial L'}$$

$$= l' - \frac{\gamma_2}{e}\frac{L'}{G'}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f + \left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\right.$$
$$\left.\times\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\}, \quad (20)$$

$$g = g' - \frac{\partial S_1}{\partial G'}$$

$$= g' + \frac{\gamma_2}{e}\frac{L'^2}{G'^2}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f + \left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\right.$$
$$\left.\times\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\}$$
$$+ \gamma_2\frac{L'^4}{G'^4}\left\{\left(-\frac{3}{2}+\frac{15}{2}\frac{H^2}{G'^2}\right)(f-l+e\sin f)+\left(\frac{9}{2}-\frac{15}{2}\frac{H^2}{G'^2}\right)\right.$$
$$\left.\times\left[\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right]\right\}, \quad (21)$$

$$h = h' - \frac{\partial S_1}{\partial H}$$

$$= h' - 3\gamma_2\frac{L'^4}{G'^4}\left[f-l+e\sin f-\frac{1}{2}\sin(2g+2f)-\frac{e}{2}\sin(2g+f)-\frac{e}{6}\sin(2g+3f)\right]\frac{H}{G'}. \quad (22)$$

In the calculation of the coordinates, l and g are not needed separately but only the sum:

$$l+g = l'+g'+\frac{\gamma_2}{e}\left(\frac{L'^2}{G'^2}-\frac{L'}{G'}\right)\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f+\left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\right.$$
$$\left.\times\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\}$$
$$+ \gamma_2\frac{L'^4}{G'^4}\left\{\left(-\frac{3}{2}+\frac{15}{2}\frac{H^2}{G'^2}\right)(f-l+e\sin f)+\left(\frac{9}{2}-\frac{15}{2}\frac{H^2}{G'^2}\right)\right.$$
$$\left.\times\left[\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right]\right\}. \quad (23)$$

Since $(1-e^2)^{-1}-(1-e^2)^{-\frac{1}{2}}$ is divisible by e^2, the first part of the difference $(l+g)-(l'+g')$ has e as a factor and not as a divisor.

4. *The second-order terms.* For use in the evaluation of F_2^* it will be convenient to put

$$\rho_2 = \frac{L'^3}{G'^3}\left[\cos(2g+2f)+e\cos(2g+f)+\frac{e}{3}\cos(2g+3f)\right], \quad (24)$$

so that

$$\frac{\partial S_1}{\partial g} = \frac{\mu^2 k_2}{L'^3}B\rho_2.$$

Also, define τ_1, τ_2 by

$$\tau_1 = \frac{1}{e}\frac{\partial \sigma_1}{\partial e} = \frac{3}{e}\frac{a^4}{r^4}\cos f - 3\frac{L^5}{G^5},$$
$$\tau_2 = \frac{1}{e}\frac{\partial \sigma_2}{\partial e} = \frac{1}{e}\left[\frac{1}{2}\frac{a^4}{r^4} - \frac{a^3}{r^3}\frac{L^2}{G^2}\right]\cos(2g+f) + \frac{1}{e}\left[\frac{5}{2}\frac{a^4}{r^4} + \frac{a^3}{r^3}\frac{L^2}{G^2}\right]\cos(2g+3f), \tag{25}$$

which are easily obtained with the aid of the expressions (4) and (5). Then

$$\frac{\partial F_{1p}}{\partial L'} = \frac{\mu^4 k_2}{L'^7}\left[-6(A\sigma_1 + B\sigma_2) + \frac{G'^2}{L'^2}(A\tau_1 + B\tau_2)\right], \tag{26}$$

$$\frac{\partial F_{1p}}{\partial G'} = -\frac{\mu^4 k_2}{L'^7}\left[\frac{G'}{L'}(A\tau_1 + B\tau_2) + \frac{3L'}{G'}\frac{H^2}{G'^2}(\sigma_1 - \sigma_2)\right]. \tag{27}$$

If the terms dependent on l are not needed to the second order in k_2, the only part of interest in (11) will be

$$\left[\frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial F_{1p}}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g}\right]_s = F_2^*,$$

where the subscript s designates the parts independent of l. The contribution to the right-hand member that has $\partial F_1^*/\partial g$ as a factor vanishes because F_1^* is a function of L', G', H' only.

The following products are easily obtained:

$$\frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 = \frac{3}{2}\frac{\mu^6 k_2^2}{L'^{10}}(A\sigma_1 + B\sigma_2)^2$$
$$\frac{\partial F_{1p}}{\partial L'}\frac{\partial S_1}{\partial l} = \frac{\mu^6 k_2^2}{L'^{10}}\left[-6(A\sigma_1 + B\sigma_2)^2 + \frac{G'^2}{L'^2}(A\sigma_1 + B\sigma_2)(A\tau_1 + B\tau_2)\right]$$
$$\frac{\partial F_{1p}}{\partial G'}\frac{\partial S_1}{\partial g} = \frac{\mu^6 k_2^2}{L'^{10}}\left[3\frac{L'}{G'}\frac{H^2}{G'^2}B(-\sigma_1\rho_2 + \sigma_2\rho_2) - \frac{G'}{L'}(AB\tau_1\rho_2 + B^2\tau_2\rho_2)\right]$$
$$\frac{\partial F_{1s}}{\partial G'}\frac{\partial S_1}{\partial g} = \frac{\mu^6 k_2^2}{L'^{10}}\left[-3\frac{L'^4}{G'^4}\left(AB + B\frac{H^2}{G'^2}\right)\rho_2\right]$$

and hence,

$$\begin{aligned} F_2^* = \frac{\mu^6 k_2^2}{L'^{10}}\Bigg\{ & A^2\left(-\frac{9}{2}\sigma_1^2 + \frac{G'^2}{L'^2}\sigma_1\tau_1\right) \\ & + AB\left[-9\sigma_1\sigma_2 + \frac{G'^2}{L'^2}(\sigma_1\tau_2 + \sigma_2\tau_1) - \frac{G'}{L'}\tau_1\rho_2 - 3\frac{L'^4}{G'^4}\rho_2\right] \\ & + B^2\left(-\frac{9}{2}\sigma_2^2 + \frac{G'^2}{L'^2}\sigma_2\tau_2 - \frac{G'}{L'}\tau_2\rho_2\right) \\ & + B\frac{H^2}{G'^2}\left[\frac{L'}{G'}(-3\sigma_1\rho_2 + 3\sigma_2\rho_2) - 3\frac{L'^4}{G'^4}\rho_2\right]\Bigg\}_s. \end{aligned} \tag{28}$$

The parts independent of l may be found by evaluating integrals of the same type as those which occurred in the preceding section. The only exceptions are the integrals

$$\frac{e}{\pi}\int_0^\pi \cos f dl = -e^2$$
$$\frac{1}{\pi}\int_0^\pi \cos 2f dl = \frac{1}{e^2}\left[2\frac{G^3}{L^3} - 3\frac{G^2}{L^2} + 1\right]$$
$$\frac{e}{\pi}\int_0^\pi \cos 3f dl = -\frac{4}{e^2}\left[2\frac{G^3}{L^3} - 3\frac{G^2}{L^2} + 1\right] + 3e^2.$$

The separate parts of F_2^* are as follows:

constant part:

$$
\begin{array}{rcllll}
-\frac{9}{2}\sigma_1^2 & = & -\frac{27}{16}\frac{L^5}{G^5} & +\frac{9}{2}\frac{L^6}{G^6} & +\frac{135}{8}\frac{L^7}{G^7} & -\frac{315}{16}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}\sigma_1\tau_1 & = & +\frac{15}{16} & -3 & -\frac{105}{8} & +\frac{315}{16} \\
\text{Sum} & = & -\frac{3}{4}\frac{L^5}{G^5} & +\frac{3}{2}\frac{L^6}{G^6} & +\frac{15}{4}\frac{L^7}{G^7} &
\end{array}
$$

$$
\begin{array}{rcllll}
-\frac{9}{2}\sigma_2^2 & = & -\frac{27}{32}\frac{L^5}{G^5} & +\frac{135}{16}\frac{L^7}{G^7} & -\frac{315}{32}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}\sigma_2\tau_2 & = & +\frac{15}{32} & -\frac{105}{16} & +\frac{315}{32} \\
-\frac{G}{L}\tau_2\rho_2 & = & +\frac{2}{3} & -\frac{5}{2} & \\
\text{Sum} & = & +\frac{7}{24}\frac{L^5}{G^5} & -\frac{5}{8}\frac{L^7}{G^7} & \\
+3\frac{L}{G}\sigma_2\rho_2 & = & -\frac{L^5}{G^5} & +\frac{5}{2}\frac{L^7}{G^7} &
\end{array}
$$

coefficient of $\cos 2g$:

$$
\begin{array}{rcllll}
-9\sigma_1\sigma_2 & = & -\frac{9}{4}\frac{L^5}{G^5} & +18\frac{L^7}{G^7} & -\frac{63}{4}\frac{L^9}{G^9} & \\
+\frac{G^2}{L^2}(\sigma_1\tau_2+\sigma_2\tau_1) & = & +\frac{5}{4} & -14 & +\frac{63}{4} & \\
-\frac{G}{L}\tau_1\rho_2 & = & +\frac{5}{4} & -\frac{21}{4} & & +2\frac{L^5}{G^5}\frac{L}{L+G} \\
-3\frac{L^4}{G^4}\rho_2 & = & & +1 & & -2 \\
\text{Sum} & = & +\frac{1}{4}\frac{L^5}{G^5} & -\frac{1}{4}\frac{L^7}{G^7} & &
\end{array}
$$

$$
\begin{array}{rclll}
-3\frac{L}{G}\sigma_1\rho_2 & = & +\frac{3}{2}\frac{L^5}{G^5} & -\frac{5}{2}\frac{L^7}{G^7} & +2\frac{L^5}{G^5}\frac{L}{L+G} \\
-3\frac{L^4}{G^4}\rho_2 & = & & +1 & -2 \\
\text{Sum} & = & +\frac{3}{2}\frac{L^5}{G^5} & -\frac{3}{2}\frac{L^7}{G^7} &
\end{array}
$$

coefficient of $\cos 4g$:

$$
\begin{array}{rcllll}
-\frac{9}{2}\sigma_2^2 & = & -\frac{9}{64}\frac{L^5}{G^5} & +\frac{9}{32}\frac{L^7}{G^7} & -\frac{9}{64}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}\sigma_2\tau_2 & = & +\frac{5}{64} & -\frac{7}{32} & +\frac{9}{64} \\
-\frac{G}{L}\tau_2\rho_2 & = & +\frac{1}{16} & -\frac{1}{16} & \\
\text{Sum} & = & 0 & & \\
+3\frac{L}{G}\sigma_2\rho_2 & = & 0 & &
\end{array}
$$

The terms with cos $4g$ disappear, and the expression for F_2^* becomes, after substitution of the appropriate expressions for A^2, AB, B^2 and BH^2/G^2,

$$F_2^* = \frac{\mu^6 k_2^2}{L'^{10}}\left[+\frac{15}{32}\frac{L'^5}{G'^5}\left(1 - \frac{18}{5}\frac{H^2}{G'^2} + \frac{H^4}{G'^4}\right) + \frac{3}{8}\frac{L'^6}{G'^6}\left(1 - 6\frac{H^2}{G'^2} + 9\frac{H^4}{G'^4}\right) - \frac{15}{32}\frac{L'^7}{G'^7}\left(1 - 2\frac{H^2}{G'^2} - 7\frac{H^4}{G'^4}\right)\right] + \frac{\mu^6 k_2^2}{L'^{10}}\left[-\frac{3}{16}\left(\frac{L'^5}{G'^5} - \frac{L'^7}{G'^7}\right)\left(1 - 16\frac{H^2}{G'^2} + 15\frac{H^4}{G'^4}\right)\right]\cos 2g', \quad (29)$$

in which g has been changed into g', which is permissible in a part that has k_2^2 as a factor.

5. *Secular and long-period terms.* The problem is now reduced to the solution of the system of canonical equations with the Hamiltonian

$$F^* = \frac{\mu^2}{2L'^2} + \frac{\mu^4 k_2}{L'^3 G'^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G'^2}\right) + F_2^*,$$

in which

$$F_2^* = F_{2s}^* + F_{2p}^*,$$

the former being a function of L', G', H only, the latter depending also on g'.

Let S^* be a new determining function,

$$S^* = L''l' + G''g' + H''h' + S_1^*(L'', G'', H'', g');$$

the equation $F^* = F^{**}$ may then be written

$$F_0^* + F_1^*\left(L'', G'' + \frac{\partial S_1^*}{\partial g'}, H''\right) + F_{2s}^* + F_{2p}^* = F_0^{**} + F_1^{**} + F_2^{**}.$$

This equation reduces to

$$F_0^* = F_0^{**},$$

$$F_1^* = F_1^{**},$$

$$\frac{\partial F_1^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + F_{2p}^* = 0, \quad (33)$$

$$F_{2s}^* = F_2^{**}. \quad (34)$$

Since

$$\frac{\partial F_1^*}{\partial G''} = \frac{3}{2}\frac{\mu^4 k_2}{L'^3 G''^4}\left(1 - 5\frac{H^2}{G''^2}\right),$$

in which L' has been written for L'', H for H'', equation (33) gives

$$\frac{\partial S_1^*}{\partial g'} = G''\gamma_2\left[\frac{1}{8}\left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4}\right)\left(1 - 16\frac{H^2}{G''^2} + 15\frac{H^4}{G''^4}\right)\right]\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\cos 2g'.$$

Substituted into

$$G' = G'' + \frac{\partial S_1^*}{\partial g'},$$

there results

$$G' = G''\left\{1 + \gamma_2\left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4}\right)\left[\frac{1}{8}\left(1 - 11\frac{H^2}{G''^2}\right) - 5\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 2g''\right\}. \quad (35)$$

Also

$$S_1^* = G''\gamma_2\left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4}\right)\left[\frac{1}{16}\left(1 - 11\frac{H^2}{G''^2}\right) - \frac{5}{2}\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g''$$

and

$$l' = l'' - \frac{\partial S_1^*}{\partial L'}$$

$$= l'' + \gamma_2\frac{L'}{G''}\left[\frac{1}{8}\left(1 - 11\frac{H^2}{G''^2}\right) - 5\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g'', \quad (36)$$

$$g' = g'' - \frac{\partial S_1^*}{\partial G''}$$

$$= g'' + \gamma_2 \left[\frac{1}{16} \frac{L'^2}{G''^2} \left(1 - 33 \frac{H^2}{G''^2}\right) - \frac{3}{16} \frac{L'^4}{G''^4} \left(1 - \frac{55}{3} \frac{H^2}{G''^2}\right) \right.$$

$$+ \left(- \frac{25}{2} \frac{L'^2}{G''^2} + \frac{35}{2} \frac{L'^4}{G''^4} \right) \frac{H^4}{G''^4} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-1}$$

$$\left. - 25 \left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4} \right) \frac{H^6}{G''^6} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-2} \right] \sin 2g'', \quad (37)$$

$$h' = h'' - \frac{\partial S_1^*}{\partial H}$$

$$= h'' + \gamma_2 \left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4} \right)$$

$$\times \left[\frac{11}{8} \frac{H}{G''} + 10 \frac{H^3}{G''^3} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-1} + 25 \frac{H^5}{G''^5} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-2} \right] \sin 2g'', \quad (38)$$

in which g' has been changed into g''.

The Hamiltonian F^{**} is a function of L', G'', H only. It is given by equations (32) and (34),

$$F^{**} = \frac{\mu^2}{2L'^2} + \frac{\mu^4 k_2}{L'^3 G''^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G''^2} \right) + F_2^{**},$$

where the last term is the first part of the right hand member of equation (29). Let n_0 be defined by

$$n_0 = \frac{\mu^2}{L'^3}$$

Then:

$$\frac{dl''}{dt} = - \frac{\partial F^{**}}{\partial L'}$$

$$= n_0 \left\{ 1 + 3\gamma_2 \frac{L'^3}{G''^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G''^2} \right) \right.$$

$$+ \gamma_2^2 \left[\frac{75}{32} \frac{L'^5}{G''^5} + \frac{3}{2} \frac{L'^6}{G''^6} - \frac{45}{32} \frac{L'^7}{G''^7} + \left(- \frac{135}{16} \frac{L'^5}{G''^5} - 9 \frac{L'^6}{G''^6} + \frac{45}{16} \frac{L'^7}{G''^7} \right) \frac{H^2}{G''^2} \right.$$

$$\left.\left. + \left(\frac{75}{32} \frac{L'^5}{G''^5} + \frac{27}{2} \frac{L'^6}{G''^6} + \frac{315}{32} \frac{L'^7}{G''^7} \right) \frac{H^4}{G''^4} \right] \right\}, \quad (39)$$

$$\frac{dg''}{dt} = - \frac{\partial F^{**}}{\partial G''}$$

$$= n_0 \left\{ 3\gamma_2 \frac{L'^4}{G''^4} \left(- \frac{1}{2} + \frac{5}{2} \frac{H^2}{G''^2} \right) \right.$$

$$+ \gamma_2^2 \left[\frac{75}{32} \frac{L'^6}{G''^6} + \frac{9}{4} \frac{L'^7}{G''^7} - \frac{105}{32} \frac{L'^8}{G''^8} + \left(- \frac{189}{16} \frac{L'^6}{G''^6} - 18 \frac{L'^7}{G''^7} + \frac{135}{16} \frac{L'^8}{G''^8} \right) \frac{H^2}{G''^2} \right.$$

$$\left.\left. + \left(\frac{135}{32} \frac{L'^6}{G''^6} + \frac{135}{4} \frac{L'^7}{G''^7} + \frac{1155}{32} \frac{L'^8}{G''^8} \right) \frac{H^4}{G''^4} \right] \right\}, \quad (40)$$

$$\frac{dh''}{dt} = -\frac{\partial F^{**}}{\partial H}$$

$$= n_0 \left\{ -3\gamma_2 \frac{L'^4}{G''^4} \frac{H}{G''} + \gamma_2^2 \left[\left(\frac{27}{8} \frac{L'^6}{G''^6} + \frac{9}{2} \frac{L'^7}{G''^7} - \frac{15}{8} \frac{L'^8}{G''^8} \right) \frac{H}{G''} + \left(-\frac{15}{8} \frac{L'^6}{G''^6} - \frac{27}{2} \frac{L'^7}{G''^7} - \frac{105}{8} \frac{L'^8}{G''^8} \right) \frac{H^3}{G''^3} \right] \right\}. \quad (41)$$

TABLE I. FACTORS ARISING IN THE ARGUMENT OF THE PERIGEE, EQUATION (36)

I	$\frac{25 \cos^6 I}{(1 - 5 \cos^2 I)^2}$	$\frac{5 \cos^4 I}{1 - 5 \cos^2 I}$
0°	1.562	−1.250
10	1.539	−1.222
20	1.476	−1.142
30	1.395	−1.023
40	1.350	−0.890
50	1.552	−0.801
55	2.140	−0.839
60	6.250	−1.202
61	10.58	−1.576
62	25.72	−2.381
63	234.76	−6.956
64	115.72	+4.716
65	12.45	+1.120
66	3.790	+0.792
70	0.232	+0.165
80	0.001	+0.005
90	0.000	0.000

This completes the solution of the problem for orbits with inclinations sufficiently far from the critical inclination. In the vicinity of the critical inclincation the results may become illusory.

The data given in Table I will serve to show that for an orbit with moderately small eccentricity, no significant loss of accuracy occurs if the inclination is even as close as 1°.5 from the critical inclination, 63°26′. The second column in this table gives the value of the factor that contains $(1 - 5 \cos^2 I)^{-2}$ as it appears in g', multiplied by $\gamma_2 e^2 (1 - e^2)^{-2}$. In h' the coefficient is $-\sqrt{5}$ times that in g'. No term of this type occurs in l'. The third column in the table indicates the magnitude of the parts of the coefficients that contain $(1 - 5 \cos^2 I)^{-1}$ as a factor. In l' and g' these numbers are multiplied by $\mp\gamma_2$ times a factor that reduces to unity for $e = 0$; in G' and h' the coefficients are diminished by a function of e that contains e^2 as a factor.

6. *Comparison with results obtained by Hill's method.* For $e = 0$, the motions obtained in equations (39), (40), (41) may be compared with my results obtained by Hill's method (Brouwer 1958). With $L'/G'' = 1$ these expressions reduce to

$$\frac{dl''}{dt} = n_0 \left[1 + \gamma_2 \left(-\frac{3}{2} + \frac{9}{2} \cos^2 I'' \right) + \gamma_2^2 \left(+\frac{39}{16} - \frac{117}{8} \cos^2 I'' + \frac{411}{16} \cos^4 I'' \right) \right],$$

$$\frac{dg''}{dt} = n_0 \left[\gamma_2 \left(-\frac{3}{2} + \frac{15}{2} \cos^2 I'' \right) + \gamma_2^2 \left(+\frac{21}{16} - \frac{171}{8} \cos^2 I'' + \frac{1185}{16} \cos^4 I'' \right) \right],$$

$$\cos I \frac{dh''}{dt} = n_0 \left[\gamma_2 \left(-3 \cos^2 I'' \right) + \gamma_2^2 \left(+6 \cos^2 I'' - \frac{57}{2} \cos^4 I'' \right) \right],$$

$$\frac{d(l'' + g'')}{dt} = n_0 \left[1 + \gamma_2 \left(-3 + 12 \cos^2 I'' \right) + \gamma_2^2 \left(+\frac{15}{4} - 36 \cos^2 I'' + \frac{399}{4} \cos^4 I'' \right) \right]. \quad (42)$$

The sidereal mean motion is

$$n_s = \frac{d(l'' + g'')}{dt} + \cos I'' \frac{dh''}{dt},$$

$$n_s = n_0 \left[1 + \gamma_2 (-3 + 9 \cos^2 I'') + \gamma_2^2 \left(+\frac{15}{4} - 30 \cos^2 I'' + \frac{285}{4} \cos^4 I'' \right) \right],$$

or

$$n_0 = n_s \left[1 + \gamma_2 (+3 - 9 \cos^2 I'') + \gamma_2^2 \left(+\frac{21}{4} - 24 \cos^2 I'' + \frac{39}{4} \cos^4 I'' \right) \right].$$

Thus

$$\frac{d(l''+g'')}{dt} = n_s\left[1 + \gamma_2(+3\cos^2 I'') + \gamma_2^2\left(+3\cos^2 I'' + \frac{3}{2}\cos^4 I''\right)\right],$$
$$\cos I''\frac{dh''}{dt} = n_s\left[\gamma_2(-3\cos^2 I'') + \gamma_2^2\left(-3\cos^2 I'' - \frac{3}{2}\cos^4 I''\right)\right]. \tag{43}$$

In the Hill method applied to the artificial satellite problem I found (Brouwer 1958, p. 435)

$$\frac{d(l''+g'')}{dt} = n_{sH}\left[1 + 3\gamma_2'\cos^2 I_H + \gamma_2'\left(\frac{21}{2}\cos^2 I_H - 30\cos^4 I_H\right)\right], \tag{44}$$

the subscript H being used with n_s, I.

The expression in square brackets is equivalent to the value of $1/(1+p)$ for $\gamma_4 = 0$ with γ_2 replaced by γ_2'. In order to compare with the result obtained above it should be noted that γ_2', $\cos I_H$ and n_{sH} differ from γ_2, H/G'' and n_s.

In the present theory, if $L' = (\mu a_0)^{\frac{1}{2}}$,

$$\gamma_2 = \frac{\mu^2 k_2}{L'^4} = \frac{k_2}{a_0^2}, \quad n_0 = \frac{\mu^2}{L'^3} = \frac{\mu^{\frac{1}{2}}}{a_0^{\frac{3}{2}}}.$$

In the Hill theory

$$\gamma_2' = \frac{k_2}{a^2}, \quad n_{sH} = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}.$$

Hence

$$\frac{\gamma_2}{\gamma_2'} = \frac{a^2}{a_0^2} = \left(\frac{n_0}{n_{sH}}\right)^{\frac{4}{3}}$$
$$= 1 + 4\gamma_2(1 - 3\cos^2 I_H),$$

from equations (42) and (44). Hence

$$\gamma_2' = \gamma_2 - 4\gamma_2^2(1 - 3\cos^2 I_H). \tag{45}$$

The inclination I_H in the Hill method was defined as the maximum latitude. In the present theory this does not apply to the inclination obtained from $\cos I'' = H/G''$ in view of the presence of the term of the first order in γ_2 and zero order in e in G/G'. For $e = 0$ the relation is

$$G = G'[1 + \tfrac{3}{2}\gamma_2 \sin^2 I \cos(2g + 2l)].$$

Then

$$\Delta\cos^2 I = -\frac{2H^2}{G'^2}\frac{\Delta G}{G'}$$
$$= -3\gamma_2 \sin^2 I\cos^2 I\cos(2g + 2l)$$

or

$$\sin^2 I = \sin^2 I''[1 + 3\gamma_2\cos^2 I''\cos(2g + 2l)].$$

Now

$$\sin^2\beta = \sin^2 I\sin^2(g + l)$$
$$= \sin^2 I''[1 + 3\gamma_2\cos^2 I''\cos(2g+2l)] \times \sin^2(g + l).$$

This is a maximum for $2g + 2l = 180°$, when

$$\sin^2\beta_{max} = \sin^2 I_H = \sin^2 I''[1 - 3\gamma_2\cos^2 I''],$$

from which

$$\cos^2 I_H = \cos^2 I''[1 + 3\gamma_2\sin^2 I'']. \tag{46}$$

The introduction of (45) and (46) into (44) yields

$$\frac{d(l''+g'')}{dt} = n_{sH}\left[1 + 3\gamma_2\cos^2 I'' + \gamma_2^2\left(\frac{15}{2}\cos^2 I'' - 3\cos^4 I''\right)\right]. \tag{47}$$

Finally

$$n_s = \frac{d(l''+g'')}{dt} + \cos I''\frac{dh''}{dt},$$
$$n_{sH} = \frac{d(l''+g'')}{dt} + \cos I_H\frac{dh''}{dt},$$

or

$$n_{sH} - n_s = (\cos I_H - \cos I'')\frac{dh''}{dt}$$
$$= \frac{3}{2}\gamma_2\sin^2 I''\cos I''\frac{dh''}{dt}$$
$$= -\frac{9}{2}n_s\gamma_2^2\sin^2 I''\cos^2 I''.$$

Introduction into (47) yields

$$\frac{d(l''+g'')}{dt} = n_s[1 + 3\gamma_2\cos^2 I'' + \gamma_2^2(3\cos^2 I'' + \tfrac{3}{2}\cos^4 I'')],$$

in agreement with the expression (43) obtained previously.

In addition to establishing the agreement between the results for $e = 0$ obtained by two different methods, the discussion serves as an illustration of the caution necessary in the comparison of results to the second power of γ_2 obtained by different treatments of the same problem.

7. *The fourth harmonic.* The coefficients of these terms are so small that it is unnecessary to consider the terms of short period. Hence the contributions to the Hamiltonian may be limited to the parts independent of l, and only the first powers are needed.

The expression for U_4 is

$$U_4 = \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right).$$

Since

$$\sin^2\beta = \frac{1}{2}\sin^2 I\,[1 - \cos(2g+2f)],$$

$$\sin^4\beta = \frac{1}{8}\sin^4 I\,[3 - 4\cos(2g+2f) + \cos(4g+4f)],$$

this becomes

$$U_4 = \frac{\mu^6 k_4}{L^{10}}\left[\left(\frac{3}{8} - \frac{15}{4}\cos^2 I + \frac{35}{8}\cos^4 I\right)\frac{a^5}{r^5} + \left(-\frac{5}{6} + \frac{20}{3}\cos^2 I - \frac{35}{6}\cos^4 I\right)\frac{a^5}{r^5}\cos(2g+2f) + \left(\frac{35}{24} - \frac{35}{12}\cos^2 I + \frac{35}{24}\cos^4 I\right)\frac{a^5}{r^5}\cos(4g+4f)\right].$$

The secular part is

$$\frac{\mu^6 k_4}{L^3 G^7}\left[\left(\frac{3}{8} - \frac{15}{4}\frac{H^2}{G^2} + \frac{35}{8}\frac{H^4}{G^4}\right)\left(\frac{5}{2} - \frac{3}{2}\frac{G^2}{L^2}\right) + \left(-\frac{5}{6} + \frac{20}{3}\frac{H^2}{G^2} - \frac{35}{6}\frac{H^4}{G^4}\right)\left(\frac{3}{4} - \frac{3}{4}\frac{G^2}{L^2}\right)\cos 2g\right].$$

The part independent of g is added to F_{2s}^*,

$$\Delta_4 F_{2s}^* = \frac{\mu^6 k_4}{L'^{10}}\left(\frac{15}{16}\frac{L'^7}{G'^7} - \frac{9}{16}\frac{L'^5}{G'^5}\right)\left(1 - 10\frac{H^2}{G'^2} + \frac{35}{3}\frac{H^4}{G'^4}\right);$$

the part having $\cos 2g$ as a factor is added to F_{2p}^*,

$$\Delta_4 F_{2p}^* = -\frac{5}{8}\frac{\mu^6 k_4}{L'^{10}}\left(\frac{L'^7}{G'^7} - \frac{L'^5}{G'^5}\right)\left(1 - 8\frac{H^2}{G'^2} + 7\frac{H^4}{G'^4}\right)\cos 2g'.$$

Now

$$\begin{aligned}
\frac{\partial}{\partial g}\Delta_4 S_1^* &= -\frac{2}{3}\frac{L'^3 G''^4}{\mu^4 k_2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\Delta_4 F_{2p}^* \\
&= \frac{5}{12}\frac{\mu^2}{L'^4}\frac{k_4}{k_2}G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left(1 - 8\frac{H^2}{G''^2} + 7\frac{H^4}{G''^4}\right)\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\cos 2g' \\
&= \frac{5}{12}\frac{\mu^2}{L'^4}\frac{k_4}{k_2}G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 2g'.
\end{aligned}$$

By integration,

$$\Delta_4 S_1^* = \frac{5}{24}\frac{\mu^2}{L'^4}\frac{k_4}{k_2}G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g'.$$

If now

$$\gamma_4 = \frac{\mu^4 k_4}{L'^8},$$

so that

$$\frac{\mu^2 k_4}{L'^4 k_2} = \frac{\gamma_4}{\gamma_2},$$

it follows that the additions to the right-hand members of equations (35) to (38) are

$$\Delta_4 G' = \frac{5}{12}\frac{\gamma_4}{\gamma_2} G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 2g'',$$

$$\Delta_4 l' = -\frac{5}{12}\frac{\gamma_4}{\gamma_2}\frac{L'}{G''}\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g'',$$

$$\Delta_4 g' = \frac{\gamma_4}{\gamma_2}\left[\frac{5}{8}\frac{L'^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right) - \frac{5}{24}\frac{L'^2}{G''^2}\left(1 - 9\frac{H^2}{G''^2}\right) + \left(-\frac{35}{3}\frac{L'^4}{G''^4} + \frac{25}{3}\frac{L'^2}{G''^2}\right)\right.$$
$$\left.\times\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + \frac{50}{3}\left(-\frac{L'^4}{G''^4} + \frac{L'^2}{G''^2}\right)\frac{H^6}{G''^6}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g'',$$

$$\Delta_4 h' = +\frac{5}{4}\frac{\gamma_4}{\gamma_2}\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\frac{H}{G''}\left[1 + \frac{16}{3}\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + \frac{40}{3}\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g''.$$

The additions to the secular motions are

$$\Delta_4\frac{dl''}{dt} = -\Delta_4\frac{\partial F_2^*}{\partial L'} = n_0\gamma_4\left[-\frac{45}{16}\left(\frac{L'^5}{G''^5} - \frac{L'^7}{G''^7}\right)\left(1 - 10\frac{H^2}{G''^2} + \frac{35}{3}\frac{H^4}{G''^4}\right)\right],$$

$$\Delta_4\frac{dg''}{dt} = -\Delta_4\frac{\partial F_2^*}{\partial G''} = n_0\gamma_4\left[-\frac{15}{16}\left(3\frac{L'^6}{G''^6} - 7\frac{L'^8}{G''^8}\right) + \frac{45}{8}\left(7\frac{L'^6}{G''^6} - 15\frac{L'^8}{G''^8}\right)\frac{H^2}{G''^2}\right.$$
$$\left.-\frac{35}{16}\left(27\frac{L'^6}{G''^6} - 55\frac{L'^8}{G''^8}\right)\frac{H^4}{G''^4}\right],$$

$$\Delta_4\frac{dh''}{dt} = -\Delta_4\frac{\partial F_2^*}{\partial H} = n_0\gamma_4\left[-\frac{5}{4}\left(3\frac{L'^6}{G''^6} - 5\frac{L'^8}{G''^8}\right)\left(3\frac{H}{G''} - 7\frac{H^3}{G''^3}\right)\right].$$

8. *The third and fifth harmonics.* Odd harmonics are to be considered only if symmetry of the earth with respect to the equator does not exist. A study of the motion of 1958 β_2 led to the introduction of these odd harmonics by O'Keefe, Eckels and Squires (1959).

The expression for U_3 is

$$U_3 = \frac{\mu A_{3.0}}{r^4}\left(\frac{5}{2}\sin^3\beta - \frac{3}{2}\sin\beta\right).$$

With

$$\sin^3\beta = \sin^3 I\left[\frac{3}{4}\sin(g+f) - \frac{1}{4}\sin(3g+3f)\right]$$

this becomes

$$U_3 = \frac{\mu A_{3.0}}{r^4}\left[\left(-\frac{3}{2}\sin I + \frac{15}{8}\sin^3 I\right)\sin(g+f) - \frac{5}{8}\sin^3 I\sin(3g+3f)\right].$$

For the sake of brevity, let $e'\sin I'$ stand for $(1 - G'^2/L'^2)^{\frac{1}{2}}(1 - H^2/G'^2)^{\frac{1}{2}}$, $e''\sin I''$ for the same expression with G' replaced by G''. The secular part may then be written

$$\Delta_3 F_{2p}^* = \frac{\mu^5 A_{3.0}}{L'^3 G'^5}e'\sin I'\left(\frac{3}{8} - \frac{15}{8}\frac{H^2}{G'^2}\right)\sin g'.$$

There is no part independent of g'.

It follows that

$$\frac{\partial}{\partial g'}\Delta_3 S_1^* = -\frac{1}{4}\frac{\mu}{L'^2}\frac{A_{3.0}}{k_2}\frac{L'^2}{G''^2}G''e''\sin I''\sin g',$$

$$\Delta_3 S_1^* = +\frac{1}{4}\frac{\mu}{L'^2}\frac{A_{3.0}}{k_2}\frac{L'^2}{G''^2}G''e''\sin I''\cos g'.$$

If

$$\gamma_3 = \frac{\mu^3 A_{3.0}}{L'^6},$$

$$\Delta_3 G' = -\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{L'^2}{G''^2} G'' e'' \sin I'' \sin g''.$$

With

$$\frac{\partial}{\partial L}(e \sin I) = \frac{\sin I}{e}\frac{G^2}{L^3},$$

$$\frac{\partial}{\partial G}(e \sin I) = -\frac{\sin I}{e}\frac{G}{L^2} + \frac{e}{\sin I}\frac{H^2}{G^3},$$

$$\frac{\partial}{\partial H}(e \sin I) = -\frac{e}{\sin I}\frac{H}{G^2},$$

the following results are easily obtained:

$$\Delta_3 l' = -\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{\sin I''}{e''}\frac{G''}{L'}\cos g'',$$

$$\Delta_3 g' = +\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{L'^2}{G''^2}\left(\frac{\sin I''}{e''} - \frac{e''}{\sin I''}\frac{H^2}{G''^2}\right)\cos g'',$$

$$\Delta_3 h' = +\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{L'^2}{G''^2}\frac{H}{G''}\frac{e''}{\sin I''}\cos g''.$$

The fifth harmonic is

$$U_5 = \frac{\mu A_{5.0}}{r^6}\left[\frac{15}{8}\sin\beta - \frac{35}{4}\sin^3\beta + \frac{63}{8}\sin^5\beta\right].$$

With

$$\sin^5\beta = \sin^5 I\left[\frac{5}{8}\sin(g+f) - \frac{5}{16}\sin(3g+3f) + \frac{1}{16}\sin(5g+5f)\right]$$

this becomes

$$U_5 = \frac{\mu A_{5.0}}{a^6}\left[\left(\frac{15}{8}\sin I - \frac{105}{16}\sin^3 I + \frac{315}{64}\sin^5 I\right)\frac{a^6}{r^6}\sin(g+f)\right.$$
$$\left. + \left(\frac{35}{16}\sin^3 I - \frac{315}{128}\sin^5 I\right)\frac{a^6}{r^6}\sin(3g+3f) + \frac{63}{128}\sin^5 I\frac{a^6}{r^6}\sin(5g+5f)\right\}.$$

$$U_5 = \frac{\mu^7 A_{5.0}}{L^{12}}\sin I\left[\frac{15}{64}\left(1 - 14\frac{H^2}{G^2} + 21\frac{H^4}{G^4}\right)\frac{a^6}{r^6}\sin(g+f)\right.$$
$$\left. - \frac{35}{128}\left(1 - 10\frac{H^2}{G^2} + 9\frac{H^4}{G^4}\right)\frac{a^6}{r^6}\sin(3g+3f) + \frac{63}{128}\left(1 - 2\frac{H^2}{G^2} + \frac{H^4}{G^4}\right)\frac{a^6}{r^6}\sin(5g+5f)\right].$$

The secular part is

$$\Delta_5 F_{2p}{}^* = \frac{\mu^7 A_{5.0}}{L'^3 G'^9} e' \sin I'$$
$$\times\left[\frac{15}{128}\left(1 - 14\frac{H^2}{G'^2} + 21\frac{H^4}{G'^4}\right)\left(7 - 3\frac{G'^2}{L'^2}\right)\sin g'\right.$$
$$\left. - \frac{35}{256}\left(1 - 10\frac{H^2}{G'^2} + 9\frac{H^4}{G'^4}\right)\left(1 - \frac{G'^2}{L'^2}\right)\sin 3g'\right].$$

Hence

$$\frac{\partial}{\partial g'}\Delta_5 S_1^* = \frac{\mu^3 A_{5.0}}{G''^5 k_2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} e'' \sin I''$$
$$\times\left[-\frac{5}{64}\left(1 - 14\frac{H^2}{G''^2} + 21\frac{H^4}{G''^4}\right)\left(7 - 3\frac{G''^2}{L'^2}\right)\sin g'\right.$$
$$\left. + \frac{35}{384}\left(1 - 10\frac{H^2}{G''^2} + 9\frac{H^4}{G''^4}\right)\left(1 - \frac{G''^2}{L'^2}\right)\sin 3g'\right],$$

$$\Delta_5 S_1^* = \frac{\mu^3 A_{5.0}}{L'^6 k_2}\frac{L'^6}{G''^6} G'' e'' \sin I''$$
$$\times\left\{\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g'\right.$$
$$\left. - \frac{35}{1152}\left(1 - \frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g'.\right.$$

With the aid of

$$\gamma_5 = \frac{\mu^5 A_{5.0}}{L'^{10}}$$

the resulting long-period terms become:

$$\Delta_5 G = \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6} G'' e'' \sin I''$$
$$\times\left\{-\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin g''\right.$$
$$\left. + \frac{35}{384}\left(1 - \frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 3g''\right\},$$

$$\Delta_5 l' = \frac{\gamma_5}{\gamma_2}\frac{L'^3}{G''^3} e'' \sin I''$$
$$\times\left\{-\frac{5}{64}\left(13 - 9\frac{G''^2}{L'^2}\right)\left(1 - \frac{G''^2}{L'^2}\right)^{-1}\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g''\right.$$
$$\left. + \frac{35}{384}\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''\right\},$$

$$\Delta_5 g' = \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6}\left[\frac{G''^2}{L'^2}\frac{\sin I''}{e''} - \frac{H^2}{G''^2}\frac{e''}{\sin I''}\right]$$
$$\times\left\{\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g''\right.$$
$$\left. - \frac{35}{1152}\left(1 - \frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''\right\}$$
$$+ \frac{\gamma^5}{\gamma_2}\frac{L'^6}{G''^6} e'' \sin I''\left\{\frac{5}{64}\left(35 - 9\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g''\right.$$
$$- \frac{35}{1152}\left(5 - 3\frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''$$
$$- \frac{15}{32}\left(7 - 3\frac{G''^2}{L'^2}\right)\frac{H^2}{G''^2}\left[3 + 16\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 40\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos g''$$
$$\left. + \frac{35}{576}\left(1 - \frac{G''^2}{L'^2}\right)\frac{H^2}{G''^2}\left[5 + 32\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 80\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos 3g''\right\},$$

$$\Delta_5 h' = \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6}\frac{H}{G''}\frac{e''}{\sin I''}$$
$$\times\left\{\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g'' - \frac{35}{1152}\left(1 - \frac{G''^2}{L'^2}\right)\right.$$
$$\left.\times\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^2}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''\right\} + \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6}\frac{H}{G''}e''\sin I''$$
$$\times\left\{\frac{15}{32}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[3 + 16\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 40\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos g''\right.$$
$$\left. - \frac{35}{576}\left(1 - \frac{G''^2}{L'^2}\right)\left[5 + 32\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 80\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos 3g''\right\}.$$

9. *Formulas for computation.* For convenience of computation the perturbations in the Keplerian elements a, e, I are given instead of those in L, G, H.

The adopted force function is

$$U = \frac{\mu}{r} + \frac{\mu k_2}{r^3}(1 - 3\sin^2\beta)$$
$$+ \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right)$$
$$+ \frac{\mu A_{3.0}}{r^4}\left(-\frac{3}{2}\sin\beta + \frac{5}{2}\sin^3\beta\right)$$
$$+ \frac{\mu A_{5.0}}{r^6}\left(\frac{15}{8}\sin\beta - \frac{35}{4}\sin^3\beta + \frac{63}{8}\sin^5\beta\right),$$

in which k_2 is a small quantity, and k_4, $A_{3.0}$, $A_{5.0}$ are assumed to be of order k_2^2.

The secular motions have been computed to $O(k_2^2)$, the coefficients of periodic terms to $O(k_2)$.

Basic constants:

a'' = semi-major axis constant
e'' = eccentricity constant
I'' = inclination constant
$n_0 = \mu^{\frac{1}{2}}a''^{-\frac{3}{2}} = 17.04337\ (a_0/R)^{-\frac{3}{2}}$ rev./day
R = equatorial radius

Abbreviations:

$$\eta = (1 - e''^2)^{\frac{1}{2}} \quad \theta = \cos I''$$
$$\gamma_2 = \frac{k_2}{a''^2} \quad \gamma_4 = \frac{k_4}{a''^4} \quad \gamma_3 = \frac{A_{3.0}}{a''^3} \quad \gamma_5 = \frac{A_{5.0}}{a''^5}$$
$$\gamma_2' = \gamma_2\eta^{-4} \quad \gamma_4' = \gamma_4\eta^{-8} \quad \gamma_3' = \gamma_3\eta^{-6} \quad \gamma_5' = \gamma_5\eta^{-10}$$

It is customary to use for the second harmonic the coefficient J; Jeffreys (1954) used for the fourth harmonic the coefficient D. The relations between J, D and γ_2, γ_4 are

$$\gamma_2 = \frac{1}{3}J\left(\frac{R}{a''}\right)^2, \quad \gamma_4 = \frac{3}{35}D\left(\frac{R}{a''}\right)^4.$$

Strictly speaking, $e'' + \delta_1 e$, $\theta' = \cos(I'' + \delta_1 I)$, $\eta' = [1 - (e'' + \delta_1 e)^2]^{\frac{1}{2}}$ should be used in the computation of the periodic terms, but since the short-period terms are obtained to $O(k_2)$, it is of no consequence if contributions of $O(k_2)$ are omitted in expressions that have γ_2 as a factor. Similarly, l'', g'' might be used in computing f', r'; but since l', g' are available, their use does not complicate the calculation.

The formulas are applicable for any eccentricity $e < 1$ and any inclination with the exception of inclinations near the critical inclination, for which $1 - 5\cos^2 I$ appears as a small divisor.

The appearance of e'' as a divisor in the short-period terms in e is apparent only. The expressions that are multiplied by e''^{-1} contain e'' as a factor, either implicitly or explicitly.

In the short-period terms in l and g a divisor e'' occurs also, but for the calculation of the position only $g + l +$ equation of the center is needed. In $g + l$ the divisor e'' is not present.

Singularities in some of the elements also occur for very small inclinations; again, no singularity is present in the coordinates. In such cases it may be found convenient to modify the formulas and obtain expressions for the perturbations in coordinates.

Secular terms:

l'' = "mean" mean anomaly

$$= n_0 t\left\{1 + \frac{3}{2}\gamma_2'\eta(-1 + 3\theta^2) + \frac{3}{32}\gamma_2'^2\eta[-15 + 16\eta + 25\eta^2 + (30 - 96\eta - 90\eta^2)\theta^2\right.$$
$$\left. + (105 + 144\eta + 25\eta^2)\theta^4] + \frac{15}{16}\gamma_4'\eta e''^2[3 - 30\theta^2 + 35\theta^4]\right\} + l_0''$$

g'' = mean argument of perigee

$$= n_0 t \left\{ \frac{3}{2}\gamma_2'(-1+5\theta^2) + \frac{3}{32}\gamma_2'^2[-35+24\eta+25\eta^2+(90-192\eta-126\eta^2)\theta^2 \right.$$

$$\left. +(385+360\eta+45\eta^2)\theta^4] + \frac{5}{16}\gamma_4'[21-9\eta^2+(-270+126\eta^2)\theta^2+(385-189\eta^2)\theta^4] \right\} + g_0''$$

h'' = mean longitude of ascending node

$$= n_0 t \left\{ -3\gamma_2'\theta + \frac{3}{8}\gamma_2'^2[(-5+12\eta+9\eta^2)\theta + (-35-36\eta-5\eta^2)\theta^3] \right.$$

$$\left. + \frac{5}{4}\gamma_4'(5-3\eta^2)\theta(3-7\theta^2) \right\} + h_0''$$

Long-period terms:

$$\delta_1 e = \left\{ \frac{1}{8}\gamma_2' e''\eta^2[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}] - \frac{5}{12}\frac{\gamma_4'}{\gamma_2'}e''\eta^2[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}] \right\}\cos 2g''$$

$$+\left\{ \frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\eta^2 \sin I'' + \frac{5}{64}\frac{\gamma_5'}{\gamma_2'\eta^2}\sin I''\,(4+3e''^2)[1-9\theta^2-24\theta^4(1-5\theta^2)^{-1}] \right\}\sin g''$$

$$-\frac{35}{384}\frac{\gamma_5'}{\gamma_2'}e''^2\eta^2 \sin I''\,[1-5\theta^2-16\theta^4(1-5\theta^2)^{-1}]\sin 3g''$$

$$\delta_1 I = -\frac{e''\delta_1 e}{\eta^2 \tan I''}$$

$$l' = l'' + \left\{ \frac{1}{8}\gamma_2'\eta^3[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}] - \frac{5}{12}\frac{\gamma_4'}{\gamma_2'}\eta^3[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}] \right\}\sin 2g''$$

$$+\left\{ -\frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\frac{\eta^3}{e''}\sin I'' - \frac{5}{64}\frac{\gamma_5'}{\gamma_2'}\frac{\eta^3}{e''}\sin I''\,(4+9e''^2)[1-9\theta^2-24\theta^4(1-5\theta^2)^{-1}] \right\}\cos g''$$

$$+\frac{35}{384}\frac{\gamma_5'}{\gamma_2'}\eta^3 e''\sin I''\,[1-5\theta^2-16\theta^4(1-5\theta^2)^{-1}]\cos 3g''$$

$$g' = g'' + \left\{ -\frac{1}{16}\gamma_2'[+(2+e''^2)-11(2+3e''^2)\theta^2-40(2+5e''^2)\theta^4(1-5\theta^2)^{-1} \right.$$

$$-400e''^2\theta^6(1-5\theta^2)^{-2}] + \frac{5}{24}\frac{\gamma_4'}{\gamma_2'}[2+e''^2-3(2+3e''^2)\theta^2-8(2+5e''^2)\theta^4(1-5\theta^2)^{-1}$$

$$\left. -80e''^2\theta^6(1-5\theta^2)^{-2}] \right\}\sin 2g'' + \left\{ \frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\left(\frac{\sin I''}{e''}-\frac{e''\theta^2}{\sin I''}\right) + \frac{5}{64}\frac{\gamma_5'}{\gamma_2'} \right.$$

$$\times\left[\left(\frac{\eta^2\sin I''}{e''}-\frac{e''\theta^2}{\sin I''}\right)(4+3e''^2)+e''\sin I''\,(26+9e''^2)\right][1-9\theta^2-24\theta^4(1-5\theta^2)^{-1}]$$

$$\left. -\frac{15}{32}\frac{\gamma_5'}{\gamma_2'}e''\theta^2\sin I''\,(4+3e''^2)[3+16\theta^2(1-5\theta^2)^{-1}+40\theta^4(1-5\theta^2)^{-2}] \right\}\cos g''$$

$$+\left\{ -\frac{35}{1152}\frac{\gamma_5'}{\gamma_2'}\left[e''\sin I''\,(3+2e''^2)-\frac{e''^3\theta^2}{\sin I''}\right][1-5\theta^2-16\theta^4(1-5\theta^2)^{-1}] \right.$$

$$\left. +\frac{35}{576}\frac{\gamma_5'}{\gamma_2'}e''^3\theta^2\sin I''\,[5+32\theta^2(1-5\theta^2)^{-1}+80\theta^4(1-5\theta^2)^{-2}] \right\}\cos 3g''$$

$$\begin{aligned}
h' = h'' + \Big\{ & -\frac{1}{8}\gamma_2' e''^2\theta[11 + 80\theta^2(1-5\theta^2)^{-1} + 200\theta^4(1-5\theta^2)^{-2}] \\
& + \frac{5}{12}\frac{\gamma_4'}{\gamma_2'} e''^2\theta[3 + 16\theta^2(1-5\theta^2)^{-1} + 40\theta^4(1-5\theta^2)^{-2}]\Big\} \sin 2g'' \\
& + \Big\{\frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\frac{e''\theta}{\sin I''} + \frac{5}{64}\frac{\gamma_5'}{\gamma_2'}\frac{e''\theta}{\sin I''}(4+3e''^2)[1 - 9\theta^2 - 24\theta^4(1-5\theta^2)^{-1}] \\
& + \frac{15}{32}\frac{\gamma_5'}{\gamma_2'} e''\theta \sin I''\,(4+3e''^2)[3 + 16\theta^2(1-5\theta^2)^{-1} + 40\theta^4(1-5\theta^2)^{-2}]\Big\} \cos g'' \\
& + \Big\{-\frac{35}{1152}\frac{\gamma_5'}{\gamma_2'}\frac{e''^3\theta}{\sin I''}[1 - 5\theta^2 - 16\theta^4(1-5\theta^2)^{-1}] \\
& - \frac{35}{576}\frac{\gamma_5'}{\gamma_2'} e''^3\theta \sin I''\,[5 + 32\theta^2(1-5\theta^2)^{-1} + 80\theta^4(1-5\theta^2)^{-2}]\Big\} \cos 3g''
\end{aligned}$$

Short-period terms included:

$$a = a''\left\{1 + \gamma_2\left[(-1+3\theta^2)\left(\frac{a''^3}{r'^3} - \eta^{-3}\right) + 3(1-\theta^2)\frac{a''^3}{r'^3}\cos(2g'+2f')\right]\right\}$$

$$\begin{aligned}
e = e'' + \delta_1 e + \frac{\eta^2}{2e''}\Big\{ & \gamma_2\left[(-1+3\theta^2)\left(\frac{a''^3}{r'^3} - \eta^{-3}\right) + 3(1-\theta^2)\left(\frac{a''^3}{r'^3} - \eta^{-4}\right)\cos(2g'+2f')\right] \\
& - \gamma_2'(1-\theta^2)[3e''\cos(2g'+f') + e''\cos(2g'+3f')]\Big\}
\end{aligned}$$

$$I = I'' + \delta_1 I + \tfrac{1}{2}\gamma_2'\theta(1-\theta^2)^{\frac{1}{2}}[3\cos(2g'+2f') + 3e''\cos(2g'+f') + e''\cos(2g'+3f')]$$

$$\begin{aligned}
l = l' - \frac{\eta^3}{4e''}\gamma_2'\Big\{ & 2(-1+3\theta^2)\left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + 1\right)\sin f' \\
& + 3(1-\theta^2)\left[\left(-\frac{a''^2}{r'^2}\eta^2 - \frac{a''}{r'} + 1\right)\sin(2g'+f') + \left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + \frac{1}{3}\right)\sin(2g'+3f')\right]\Big\}
\end{aligned}$$

$$\begin{aligned}
g = g' + \frac{\eta^2}{4e''}\gamma_2'\Big\{ & 2(-1+3\theta^2)\left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + 1\right)\sin f' \\
& + 3(1-\theta^2)\left[\left(-\frac{a''^2}{r'^2}\eta^2 - \frac{a''}{r'} + 1\right)\sin(2g'+f') + \left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + \frac{1}{3}\right)\sin(2g'+3f')\right]\Big\} \\
& + \tfrac{1}{4}\gamma_2'\{6(-1+5\theta^2)(f' - l' + e''\sin f') \\
& + (3-5\theta^2)[3\sin(2g'+2f') + 3e''\sin(2g'+f') + e''\sin(2g'+3f')]\}
\end{aligned}$$

$$\begin{aligned}
h = h' - \tfrac{1}{2}\gamma_2'\theta[& 6(f' - l' + e''\sin f') - 3\sin(2g'+2f') \\
& - 3e''\sin(2g'+f') - e''\sin(2g'+3f')].
\end{aligned}$$

f', r' are to be computed from

$$E' - e''\sin E' = l'$$

$$\tan\tfrac{1}{2}f' = \left(\frac{1+e''}{1-e''}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}E' \qquad\qquad \frac{r'}{a''}\sin f' = (1-e''^2)^{\frac{1}{2}}\sin E'$$

$$\frac{a''}{r'} = \frac{1+e''\cos f'}{1-e''^2} \qquad \text{or} \qquad \frac{r'}{a''}\cos f' = \cos E' - e''$$

$$\frac{r'}{a''} = 1 - e''\cos E'$$

For the calculation of the coordinates at any time the complete values of e and l should be used for the solution of Kepler's equation,

$$E - e \sin E = l$$

and subsequently r and f, which may then be used in the formulas:

$$x = r[\cos(g+f)\cos h - \sin(g+f)\sin h\cos I]$$
$$y = r[\cos(g+f)\sin h + \sin(g+f)\cos h\cos I]$$
$$z = r\sin(g+f)\sin I$$

A convenient alternative form is:

$$x = A_x(\cos E - e) + B_x \sin E$$
$$y = A_y(\cos E - e) + B_y \sin E$$
$$z = A_z(\cos E - e) + B_z \sin E$$

$$A_x = a[\cos g\cos h - \sin g\sin h\cos I]$$
$$B_x = -a(1-e^2)^{\frac{1}{2}}[\sin g\cos h + \cos g\sin h\cos I]$$
$$A_y = a[\sin g\cos h\cos I + \cos g\sin h]$$
$$B_y = a(1-e^2)^{\frac{1}{2}}[\cos g\cos h\cos I - \sin g\sin h]$$
$$A_z = a\sin g\sin I$$
$$B_z = a(1-e^2)^{\frac{1}{2}}\cos g\sin I$$

Noted added in proof. The lack of uniformity in notation of the coefficients of the second and fourth harmonics of the earth's potential in papers dealing with the motion of artificial satellites calls for a comment on this subject.

The table below contains a listing of some of the designations used and their relations to the coefficients B_p in the expression of the force function of a body with rotational symmetry,

$$F = \frac{\mu}{r}\left[1 + \sum_{p=2}^{\infty}\frac{B_p P_p(\sin\beta)}{r^p}\right],$$

in which P_p are Legendre polynomials and $\mu = GM$. The expression is an adaptation of the Laplacian expression given by Tisserand.

In addition to the equivalents of B_2 and B_4 the table gives those of the ratio B_4/B_2^2, which is unity for the special case treated by Vinti (1959), in which the terms with small divisors near the critical inclination vanish. No effort has been made to make the tabulation complete.

Laplace	B_2	B_4	B_4/B_2^2	Tisserand, *Méc. Cél.* II, 320, 1890
H. Struve	$-\frac{2}{3}k$	$\frac{2}{3}l$	$\frac{3}{2}l/k^2$	*Suppl. I, Obs. Pulkovo* 1888
W. de Sitter	$-\frac{2}{3}JR^2$	$\frac{4}{15}KR^4$	$\frac{3}{5}K/J^2$	*B. A. N.* 2, 97, 1924
D. Brouwer	$-2k_2$	$\frac{8}{3}k_4$	$\frac{2}{3}k_4/k_2^2$	*A. J.* 51, 223, 1946
H. Jeffreys	$-\frac{2}{3}JR^2$	$\frac{8}{35}DR^4$	$\frac{18}{35}D/J^2$	*M. N.* 14, 433, 1954
Y. Kozai	$-\frac{2}{3}A_2$	$\frac{8}{35}A_4$	$\frac{18}{35}A_4/A_2^2$	
P. Herget and P. Musen	$-2k_2$	$8k_4$	$2k_4/k_2^2$	*A. J.* 63, 430, 1958
J. O'Keefe et al.	$+A_{2,0}/\mu$	$+A_{4,0}/\mu$	$\mu A_{4,0}/A_{2,0}^2$	*A. J.* 64, 235, 1959
B. Garfinkel	$-2k$	k'	$\frac{1}{4}k'/k^2$	This issue
J. Vinti	$-J_2R^2$	$-J_4R^4$	J_4/J_2^2	*J. of Res. Nat. Bureau of Standards* 62B, 105, 1959

In the table R represents the earth's equatorial radius. Ignoring the presence of R^2 and differences in sign, essentially three different coefficients for the second harmonic have been used in recent papers. For the coefficients of the fourth harmonic six different choices are listed. I now regret that I introduced k_2, k_4 in my paper in 1946. The principal reason was that they give a particularly simple form for the expression of the potential in the equatorial plane. If I could have foreseen the increase in interest in the subject and the confusion to which I was contributing, I would have chosen the coefficients B_p or the alternative form

$$F = \frac{\mu}{r}\left[1 - \sum_{p=2}^{\infty} J_p\left(\frac{R}{r}\right)^p P_p(\sin\beta)\right],$$

which was used by Vinti (1959). I intend to revert to this form and recommend this to other authors.

10. *Acknowledgments*. The investigation contained in this article was supported by a contract with the Air Force Cambridge Research Center, AF 19(604)4137. Before this contract was negotiated, Dr. Boris Garfinkel, at the Army Ordnance Ballistic Research Laboratory at Aberdeen Proving Grounds, Maryland, and I had been working independently on different solutions of the same problem. We soon decided that the best policy would be for each of us to continue our work independently, but to make comparisons of our results. Dr. Garfinkel's results are presented in a separate article in the same issue of this Journal. The effects of the odd harmonics, presented in Section 8, were not included by Garfinkel. With this exception, the two sets of results should be identical.

Anyone who has ever carried out developments of the type presented here knows that extreme caution is necessary if errors are to be avoided, even if the method used is straightforward. The comparison with Dr. Garfinkel's results at various stages therefore provided useful checks. In addition, I wish to record my indebtedness to Dr. Gen-ichiro Hori who made a critical study of the entire manuscript. Owing to his help I am confident that there are no errors in the results, except possibly typographical errors. Every effort has been made to eliminate these also.

Dr. Hori has undertaken the study of orbits near the critical inclination. It is his intention to present his results in a future issue of the Journal.

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PHOTOGRAPHIC DIMENSIONS OF THE BRIGHTER GALAXIES

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Abstract. Standard photometric dimensions of 37 galaxies are derived from recent detailed photometric studies. The relations of photometric dimensions to the laws of luminosity distribution in galaxies of different types are discussed.

Micrometric dimensions of over 1200 Shapley-Ames galaxies (to be published later) have been derived from 13 main series of photographic determinations, including 6 series of new data. The relations between systems of micrometric diameters D and axial ratios $R = d/D$ are determined and the relations between photometric and micrometric dimensions are discussed.

A provisional reduction of 10 series to the (corrected) system of the Heidelberg survey by Reinmuth was made for the Shapley-Ames objects. The probable errors of the reduced values from individual series are of the order of 12 per cent for both D and R.

Strong correlations between apparent diameter and axial ratio, i.e., tilt, and galactic latitude are indicated. The reduced or "face-on" diameters $D(0)$ of galaxies of a given type are also closely correlated with total magnitude m_T and can be used as distance indicators. An application is made to the calibration in terms of the current distance scale of the relative distances of several clouds and clusters of bright southern galaxies observed in the Mount Stromlo survey.

The application of the corrected mean diameters and revised luminosity laws to the determination of improved incompleteness corrections and total (or asymptotic) magnitudes is briefly discussed.

In the preparation of a reference catalogue of bright galaxies based on the Harvard "Survey of Galaxies brighter than the 13th magnitude" (Shapley and Ames 1932, de Vaucouleurs 1953b) auxiliary studies have been made of the magnitude and dimension systems used in various published and unpublished surveys in both hemispheres. The discussion of photographic magnitudes published three years ago (de Vaucouleurs 1956a, 1957a) has indicated the need for a detailed study of dimension data. The present paper summarizes the main results of this study and discusses briefly some applications; further details will be found in *Annales de l'Observatoire du Houga*, Vol. **II**, Part 2. The revised reference catalogue is now being prepared for publication.

1. *Photometric dimensions*. The apparent dimensions of galaxies can be precisely defined and correctly determined through a detailed photometry of the luminosity distribution in the nebular image. The *brightness dimensions* $D(m_1)$, $d(m_1)$ refer to a given isophote (or best fitting ellipse) of specified surface brightness, for instance $m_1 = 25.0$ mag/sec² (pg) (Redman 1936; Hazen 1957).

TABLE I. PHOTOMETRIC DIMENSIONS OF 37 GALAXIES

NGC	Type	D_e	d_e/D_e	D_1*	d_1/D_1*	Sources
LMC	SB(s)m	5°5	0.89	—	—	1
SMC	SB(s)mp	1°8	0.5:	—	—	1
224	SA(s)b	1°2	0.30	3°05	0.29	2
598	SA(s)c	24′2	0.66	58′ :	0.62	3
1291	SB(s)O+	7′5	0.55	—	—	4
1313	SB(s)d	4′0	0.9:	—	—	5
1316	SOp	3′1	0.71	—	—	5, 6
3115	E+7	3′1	0.32	13′4	0.5:	7, 8
3379	EO	1′1	1.0:	—	—	8
4267	SB(s?)O−	—	—	2′7	0.96	9
4350	SAO sp.	1′5	0.50	3′2	0.62	9
4365	E3	2′8	0.75	6′7	0.74	9
4374	E+1	3′0	0.90	7′1	0.99	9
4406	E+3	6′6	0.70	(12′3)	0.60	9
4417	SO	1′6	0.45	3′5	0.49	9
4425	SO sp.	—	—	2′7	0.40	9
4442	SB(s)O°	—	—	4′6	0.40	9
4459	SA(r)O+	—	—	4′2	0.83	9
4461	SB(s)O+:	—	—	3′5	0.40	9
4472	E2	—	—	12′6	0.80	9
4473	E5	1′9	0.60	4′7	0.63	9
4477	SB(s?)O°:	—	—	3′6	0.90	9
4486	E1p	2′8	0.88	9′1	0.75	6, 9
4494	E1	1′2	0.9:	5′5	0.9:	7, 8
4503	SBO−	—	—	3′5	0.47	9
4526	SAB(s:)O°	—	—	7′5	0.37	9
4552	EO	1′9	0.90	5′6	0.87	9
4564	SO?	1′6	0.50	3′0	0.54	9
4570	SO sp.	—	—	3′8	0.33	9
4578	SO?	—	—	3′1	0.70	9
4594	SA(s:)a	4′6	0.50	—	—	10
4621	E5	3′0	0.70	5′7	0.82	9
4649	E2	3′0	0.80	9′2	0.78	9
4694	SBOp	—	—	3′4	0.57	9
4754	SB(r:)O−	—	—	4′6	0.53	9
5643	SA(s:)c	2′3	1.00	—	—	5
6744	SAB(r)bc	9′ :	0.61	—	—	5

* $m_1 = 25.0$ mag/sec² (pg).

Sources:

1 = de Vaucouleurs 1957b
2 = de Vaucouleurs 1958
3 = de Vaucouleurs 1959
4 = de Vaucouleurs 1956c
5 = unpublished
6 = Sersic 1958
7 = Oort 1940
8 = de Vaucouleurs 1953a
9 = Hazen 1957
10 = de Vaucouleurs 1948

The *effective dimensions* D_e, d_e refer to the isophote (or best fitting ellipse) within which is emitted half the total (photographic) luminosity of the nebula (de Vaucouleurs 1948, 1953a).

The determination of brightness dimensions requires absolute measurements of surface brightness on a given magnitude scale in mag/sec²; these dimensions vary with the particular brightness level arbitrarily selected in their definition. The determination of effective dimensions requires only relative measurements of surface intensities with arbitrary zero point; the dimensions are, in principle, intrinsic properties of the nebula, independent of the observational circumstances (except, of course, wave length of light).

Both types of measurements are slow and tedious so that reliable data on either brightness or effective dimensions are at present available for only a few dozen objects listed in Table I. More work along these lines is greatly needed.

On the other hand, "maximum" dimensions $D_{\max}$, $d_{\max}$ measured on microphotometer tracings are now available for several hundred bright galaxies observed under fairly homogeneous conditions; these values are approximate brightness dimensions relative to an unspecified mean (photographic) brightness threshold, usually fainter than 25 mag/sec². Comparison with calibrated photometric data determined this threshold to be about 25.3 mag/sec² (Patterson 1941) and 26.7 mag/sec² (Holmberg 1958).

All photometric dimensions which are in the nature of brightness dimensions are determined by the threshold brightness B_1 and the law of luminosity distribution $B(r)$ in the image of the galaxy, while the effective dimensions depend only on the latter. The ratio of brightness dimensions relative to different magnitude thresholds $m_1, m_2, \cdots$ depends on the luminosity laws; experience shows that, in practice, only two main groups need be considered in a discussion of apparent diameters: 1) the ellipticals, 2) the lenticulars and spirals.

The first group obeys closely the luminosity law $\log B = ar^{1/4} + b$ (de Vaucouleurs 1948, 1953a, 1956a), which may be written in the reduced form

$$\log \mathcal{B} = -3.33(\alpha^{1/4} - 1) \tag{1}$$

if $\mathcal{B} = B/B_e$, $\alpha = r/r_e$, r_e, B_e being the effective radius and corresponding surface brightness.

The second group follows approximately, except near the center, an exponential law, $\log B = A - Kr$ (Patterson 1940, de Vaucouleurs 1958), which similarly can be written in the reduced form

$$\log \mathcal{B} = -0.729\,(\alpha - 1) \tag{2}$$

with $\mathcal{B}$ and α defined as above.

There is *a priori* no constant relation between brightness dimensions and effective dimensions since it depends on B_e and luminosity law; however, if the dwarf ellipticals and irregulars are excluded, the mean surface brightness of the majority of catalogued galaxies is restricted to a fairly small range (cf. de Vaucouleurs 1957a, Fig. 19 and sections 5 and 6 below) so that, for a given type, a statistical relation exists between

these dimensions. For the small sample in Table I brightness dimensions relative to $m_1 = 25.0$ mag/sec^2 are, on the average, about 2.5 times the effective dimensions: $D(m_1)/D_e \simeq 2.5$. It is then possible to estimate roughly, by means of equations (1) and (2), the variation of brightness diameter corresponding to a given variation Δm_1 of the threshold brightness when $m_1 \simeq 25$; for $\Delta m_1 = \pm 0.25$ mag, $\Delta \log \alpha = \pm 0.04$, or 10 per cent on D_1, for ellipticals, and $\Delta\alpha = \pm 0.14$, or 5 per cent on D_1, for spirals. The exponential luminosity law in the outer parts of spirals makes their diameters less sensitive to fluctuations in the brightness threshold; this is confirmed by a direct comparison of micrometric diameters (section 4).

2. *Micrometric dimensions*. For the great majority of galaxies the only dimension data available are direct micrometer measurements of photographic negatives. These micrometric dimensions refer to an unspecified mean brightness threshold depending in an *a priori* unknown manner on the properties of the object (size, shape, luminosity gradient), of the foreground (airglow, galactic and zodiacal light), of the photographic plate (exposure, graininess, contrast factor) and of the observer and observing conditions. However, when the plates are secured and measured under homogeneous conditions, micrometric dimensions refer to a relatively well-defined brightness threshold which can be calibrated by appropriate comparisons with photometric data.

The main purpose of the present paper is to investigate the systematic errors to which micrometric dimensions are subject, to obtain the reduction formulae to a standard mean system and to calibrate the mean system of brightness dimensions so defined.

The material available for a study of micrometric dimensions of bright galaxies is summarized in Table II. Altogether over 30 lists and catalogues grouped in 13 main series were used; series IV, V, VI, VII, VIII, IX include new material obtained from the writer's examination of the following plate collections: (IV) Isaac Roberts 20-inch, now at the Paris Observatory, (V, Va) Mount Stromlo 30-inch Reynolds Survey (de Vaucouleurs 1956b) and 74-inch survey (de Vaucouleurs 1960), (VI) Lick 36-inch Crossley (modern plates only), (VII) and (VIII) Mount Wilson 60-inch and 100-inch, (IX) Palomar 200-inch. In series (Va), (VII), (VIII) and (IX) two sets of dimensions are available for each galaxy: D_i, d_i for the inner bright regions and D_o, d_o for the outer, fainter regions (cf. de Vaucouleurs 1956b).

After some experiment it was decided that the significant parameters for a comparison of different lists were the major diameter D and the axial ratio $R = d/D$ (not D and d separately; see also Holmberg 1946). Further, the discussion was limited to objects listed in the Shapley-Ames catalogue, which offers the greatest overlap between the various series.

The comparison of the dimension systems D_1, D_2 of two catalogues was made graphically on logarithmic correlograms (Fig. 1 right), and because it includes the largest number of objects north of declination $-20°$, the Heidelberg survey (series Ia, Reinmuth 1926) was taken as the standard for comparison purposes. In general an equation of the form

$$D_2 = AD_1 \tag{3}$$

suffices to represent the relation between series I and any other series (II, III, IV, V, VI, etc.) of dimensions secured with small telescopes. However, when the comparison is with dimensions given by the larger telescopes (series VII, VIII, IX etc.) an equation of the form

$$D_2 = AD_1 + B \tag{4}$$

is often required. The reason for this difference will be given in section 3. Experience shows that ellipticals on the one hand, lenticulars and spirals on the other, give different coefficients in (3) or (4) as could be expected from their different luminosity laws (section 1). In a few cases a non-linear relation was indicated, pointing to a faulty dimensions system in some series, especially for ellipticals (series IIb, XIb).

The comparison of axial ratios was made similarly on logarithmic plots (Figure 1 left). In general, two groups of series are indicated: (a) series for which the axial ratios R are nearly on the same system as that of the Heidelberg survey (series Ia) whose systematic errors have been determined by Holmberg (1946); and (b) series for which the axial ratios are in agreement with those determined by photometry and whose systematic errors are consequently small or nil. This grouping was found by comparing the empirical (R_1, R_2) relation to the relation predicted by Holmberg's corrections. A relation distinctly different from that found in either (a) or (b) was observed in only a few cases (series VIII, XII, XIIIb).

TABLE II. SOURCES OF MICROMETRIC DIMENSIONS OF GALAXIES

Series	Observatory and telescope	Scale 1 mm =	Number of galaxies Total	Number of galaxies In HA88, 2	Sources
Ia	Heidelberg, 16″	1′.7	3500±	776	1
Ib	—	—	827	149	2
Ic	—	—	4866	426	3
IIa	Harvard-Boyden, 24″	1′.0	243	243	4
IIb	— —	—	486	486	5
IIIa	Helwan, 30″	0′.9	58	40	6
IIIb	—	—	184	91	7
IIIc	—	—	169	60	8, 9
IIId	—	—	461	65	10
IV	Roberts, 20″	1′.37	600	289	11
V	Mt. Stromlo, 30″	1′.12	460	205	12
Va	Mt. Stromlo, 74″	0′.38	47	40	13
Vb	Pretoria, 74″	0′.38	34	19	11
Vc	Harvard-Boyden, 60″	0′.44	101	61	11
VIa	Lick, 36″	0′.63	494	105	11
VIb	—	—		54	11
VIc	—	—		59	11
VIIa	Mt. Wilson, 60″	0′.45	384	68	11
VIIb	—	—		166	11
VIIc	—	—		53	11
VIIIa	Mt. Wilson, 100″	0′.27	503	51	11
VIIIb	—	—		182	11
VIIIc	—	—		167	11
IXa	Mt. Palomar, 200″	0′.20	262	118	11
IXb	—	—		57	11
X	Lick, 36″	0′.63	477	257	14
XIa	Mt. Wilson, 60″	0′.45	101	29	15
XIb	Lick, 36″ Mt. Wilson, 60″, 100″	0′.63 0′.45, 0′.27	400	303	16
XII	Franklin-Adams, 10″ Lick, 36″	3′.0 0′.63	122	96	17
XIIIa	Helwan, 30″	0′.9	202	162	18
XIIIb	Lick, 36″	0′.63			
XIIIc	Mt. Wilson, 60″, 100″	0′.45, 0′.27			

Sources:
1 = Reinmuth 1926
2 = Holmberg 1937
3 = Reiz 1941
4 = Shapley-Ames 1932, "authority (a)" only
5 = Shapley 1934
6 = Knox-Shaw 1912, 1915
7 = Gregory 1920, 1921
8 = Know-Shaw 1924
9 = Madwar 1935
10 = Svenonius 1938
11 = de Vaucouleurs unpublished
12 = de Vaucouleurs 1956b
13 = de Vaucouleurs 1960
14 = Curtis 1918
15 = Pease 1917, 1920
16 = Hubble 1926
17 = Lundmark 1927
18 = Danver 1942

The coefficients A and B of equations (3) and (4) and the axial ratio group (a or b) are listed in Table III; both the raw empirical values and the adjusted values resulting from the discussion in section 3 are given. The scale coefficient A (for the series taken with the smaller telescopes) is correlated with the mean epoch of the plates; prior to 1910–20 $\bar{A} \simeq 0.9$ (E) and 0.95 (S) as against $\bar{A} \simeq 1.3$ (E) and 1.15 (S) since 1920–30; this is clearly the effect and measure of increased plate sensitivity and fuller exposure.

The sum $A + B$, measuring the diameter D_2 which corresponds to $D(Ia) = 1'.0$, is correlated with the plate scale (series earlier than 1920 are excluded); a galaxy of diameter 1′.0, as measured on small scale plates (0′.5 to 1′.0/mm) is 1′.5 on medium-scale plates (2′ to 3′/mm) and 2′ or more on large-scale plates (4′/mm and over).

This variation is brought about by the relation between linear image size and threshold brightness or, in other words, by the signal/noise ratio associated with the graininess of the photographic image, as discussed in the following section.

3. *Relation between photometric and micrometric dimensions.* The threshold brightness is essentially determined by the smallest perceptible excess of photographic density above background that can be significantly associated with the nebular image; this significance is judged by comparison with the density fluctuations of the plate background determined by the graininess of the photographic image. Since the relative density fluctuations increase as the scanned area decreases, the density excess that can be considered as significant increases also and in consequence the threshold brightness must vary with image size. This is confirmed by observation; for example, Figure 2 shows the variation of threshold surface brightness as a function of the diameter of uniform circular images as empirically determined by Hubble (1932). The threshold brightness is independent of image size only when the diameter is 1 cm or greater; for a 1 mm image it is about 1 mag brighter.

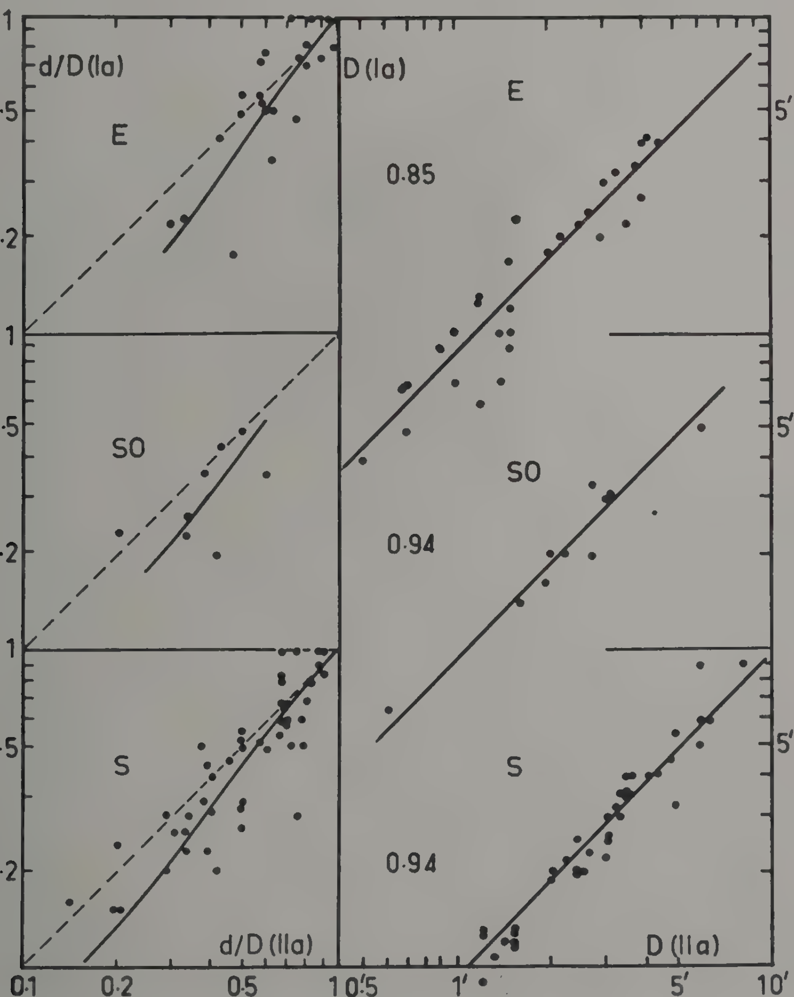

Figure 1. Comparisons of micrometric dimension systems: Ia (Reinmuth 1926) and IIa (Shapley and Ames 1932) Correlation between diameters D (right): the scale ratio $D(\mathrm{Ia})/D(\mathrm{IIa})$ is given for E, SO, and S objects. Correlation between axial ratios $R = d/D$ (left): the curve represents the relation according to Holmberg (1946) between Reinmuth's data and the true (photometric) axial ratio.

TABLE III. RELATIONS BETWEEN MICROMETRIC DIMENSION SYSTEMS

Series	Ellipticals: A + B Empirical	Ellipticals: A + B Adjusted	Spirals: A + B Empirical	Spirals: A + B Adjusted	Axial ratio E	Axial ratio S	Notes
Ia	1.00	1.00	1.00	1.00	Ia	Ia	Std.
Ib	1.00	1.00	1.00	1.00	Ia	Ia	
Ic	0.80	0.85 − 0′.05	0.90	0.92 − 0′.05	*	*	(1)
IIa	1.18	0.95 + 0′.2	1.07	0.9 + 0′.15	ptm.	ptm.	
IIb	*	—	1.43	1.25 + 0′.35	ptm.	ptm.	(2)
III	—	—	1.0:	0.85 + 0′.15	—	Ia	
IV	1.05:	0.95 + 0′.1	1.00	0.95 + 0′.1	Ia	Ia	
V D_i	0.94	0.85 + 0′.1	0.84	0.80 + 0′.05	ptm.	ptm.	(3)
— D_0	1.40	1.2 + 0′.3	1.20	1.1 + 0′.2	ptm.	ptm.	
Va D_i	1.16:	0.8 + 0′.3	0.83	0.65 + 0′.2	ptm.?	ptm.	
— D_0	1.5: + 1′.0:	1.4 + 0′.6	1.33	1.2 + 0′.5	ptm.?	ptm.	
VI	1.07:	0.9: + 0′.3	1.06	0.85 + 0′.2	Ia?	Ia	
VII D_i	0.84	0.6 + 0′.2	0.82	0.6 + 0′.2	Ia	Ia	
— D_0	1.4: + 0′.3:	1.2 + 0′.5	1.10 + 0′.7	1.1 + 0′.4	Ia	Ia	
VIII D_i	0.6:	0.6 + 0′.2	0.82	0.6 + 0′.2	ptm.?	Ia	
— D_0	1.4: + 0′.4	1.3 + 0′.6	1.10 + 0′.7	1.15 + 0′.5	ptm.?	Ia	
IX D_i	0.8 + 0′.2	0.7 + 0′.3	0.8 + 0′.3	0.8 + 0′.35	ptm.?	Ia?	
— D_0	1.8 + 1′.0	1.9 + 0′.9	1.3 + 1′.2	1.4 + 0′.7	Ia?	Ia	
X	0.77	—	0.95	—	Ia	Ia	
XIa	0.9:	—	1.0:	—	Ia	Ia	
XIb	*	—	0.95	—	ptm.	Ia	(4)
XII	0.82	—	0.95	—	*	Ia	(5)
XIIIa	—	—	1.15	—	—	Ia?	
XIIIb	—	—	1.10	—	—	*	(6)
XIIIc	—	—	1.00	—	—	Ia	

Notes:

(1) after Holmberg's discussion (1946) $R(\mathrm{Ic}) = R(\mathrm{Ia}) - 0.02$.
(2) the D scale for E is non-linear: $\log D(\mathrm{Ia}) = 0.86 \log D(\mathrm{IIb}) - 0.17$.
(3) coefficients for D_i inferred from $\overline{D_i/D_0} = 0.67$ (E) or 0.70 (S) (de Vaucouleurs 1956b).
(4) the D scale for E is non-linear: $\log D(\mathrm{Ia}) = 0.75 \log D(\mathrm{XIb}) + 0.10$.
(5) the R scale for E is irregular: $R(\mathrm{Ia}) = R(\mathrm{XII})$ for $R > 0.7$,
$= R(\mathrm{XII}) + 0.10$ for $R \leq 0.7$.
(6) the R scale for S is irregular: $R(\mathrm{Ia}) = R(\mathrm{XIIIb})$ for $R > 0.6$,
$\doteq R(\mathrm{XIIIb}) + 0.08$ for $R \leq 0.6$.

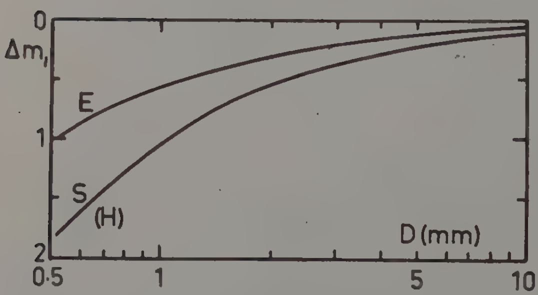

Figure 2. Variation of threshold surface brightness as a function of image diameter for ellipticals and spirals from equation (7) and for uniform images after Hubble (1932).

A similar effect, perhaps with a different amplitude, applies to the threshold brightness m_1 that defines the visible edge of a nebular image; when the linear size (in millimeters on the plate) is large, and for a given set of observational circumstances, m_1 depends only or primarily on the density gradient, i.e., on the luminosity law, in the nebular image; in other words, for a given galaxy type, m_1 is a constant, characteristic of the micrometric dimensions system which then approximates a photometric system.

On the other hand, when the linear size is small the threshold brightness m_1 varies with image size as in Figure 2 so that micrometric dimensions measured on small scale photographs do not

approximate any given system of photometric dimensions; the measurements of small galaxies come out too small compared with the system of brightness dimensions defined by the larger galaxies.

It can be shown that this effect is measured by the additive constant B in equation (4). For example, the relation between the micrometric dimensions D in the Heidelberg survey (series Ia, scale $1'.7$/mm) and the brightness dimensions at $m_1 = 25.0$ mag/sec² are of this form:

$$D_1(m_1{=}25) = 2.4\,D + 1'.2 = 2.4\,(D + 0'.5) \qquad \text{(E)},$$

$$D_1(m_1{=}25) = 1.25\,D + 0'.6 = 1.25\,(D + 0'.5) \qquad \text{(S)}.$$

Thus the brightness diameter D_1 relative to a constant threshold m_1 is of the form

$$D_1 \propto (D + \delta), \tag{5}$$

where δ is a small constant. The relative error in D is then given by the expression

$$\Delta \log D = \log D_1/D = \log\,(1 + \delta/D). \tag{6}$$

The corresponding variation of the threshold brightness is

$$\Delta m_1 = K \log\,(1 + \delta/D) \tag{7}$$

if K is the slope of the relation $m_1 = f(\log D)$. This slope can be computed for the luminosity laws (1) and (2) if it is assumed, from the data in Table I, that $\alpha \simeq 1$ (E) or 2 (So, S) for $A = 1$ (see section 6). The mean slope near $A = 1$ is then $K \simeq 5$ (E) or 9 (So, S); the values of Δm_1 so computed are shown in Figure 2 where the curve for spirals accidentally coincides with Hubble's empirical relation. The similarity of the relations $\Delta m_1 = f(D)$ with Hubble's confirms the interpretation of the constant B in (4).

This result makes it possible to adjust the coefficients of the empirical relations between two systems of micrometric dimensions. If the general relation between brightness and micrometric diameters is

$$D(m_1) = aD + b = a(D + \delta) \tag{8}$$

and if the relative error in D is, for a given galaxy type, the same function of the linear size of the image on the plate for all series, δ must be proportional to the plate scale S in minutes of arc per millimeter, i.e.,

$$\delta = S\delta_0. \tag{9}$$

It follows that the relation between two systems of micrometric diameters is

$$a_1(D_1 + S_1\delta_0) = a_2(D_2 + S_2\delta_0)$$

or

$$D_1 = \frac{a_2}{a_1} D_2 + \delta_0 \left(\frac{a_2}{a_1} S_2 - S_1\right) \tag{10}$$

which is of the form (4). The values $\delta = 0'.5$ for series Ia ($S = 1'.7$/mm) suggest that $\delta_0 = 0'.3$ for all galaxy types and this is confirmed by the satisfactory fit of the empirical correlograms given by the adjusted coefficients based on this value of δ_0, as listed in Table III. It is clear, however, that very small values of D, say less than about 3δ or 1 mm on the plates, are of lower than average accuracy.

The objects in Table I whose luminosity profiles are known on an absolute scale have been used for a provisional calibration of the scale coefficient A. A final calibration will require more standards, especially for spirals, than are presently available. The mean surface brightness thresholds corresponding to the micrometric major diameters in several series are listed in Table IV. Two values are given: m_1 is the uncorrected

TABLE IV. MEAN THRESHOLD BRIGHTNESS OF MICROMETRIC DIAMETER SYSTEMS

Series	Ellipticals m_1	m_1'	n	Lenticulars m_1	m_1'	n	Mean diameter E	So
Ia	22.85	23.2	9	23.74	24.15	15	$2'.9$ = 1.7 mm	$2'.7$ = 1.6 mm
Ib	22.65	23.0	4	23.90	24.8	8	$3'.1$ = 1.8 —	$2'.0$ = 1.2 —
Ic	22.31	22.7	7	22.86	23.85	12	$2'.4$ = 1.4 —	$2'.0$ = 1.2 —
IIb	—	—		25.62	25.9	14	—	$4'.4$ = 4.4 —
IV	22.62	23.0	4	23.04	23.75	6	$2'.5$ = 1.8 —	$2'.2$ = 1.6 —
VI	22.38	22.6	5	23.44	23.7	5	$1'.9_5$ = 3.1 —	$2'.5$ = 4.0 —
VII D_0	—	—		24.70	24.8	4	—	$4'.5$ = 10 —
VIII D_i	21.13:	21.3:	2	22.01	22.25	6	$0'.9$ = 3.3 —	$1'.1$ = 4.1 —
— D_0	24.08:	24.1:	2	24.67	24.75	5	$3'.8$ = 14 —	$3'.4$ = 12.5 —
IX D_i	22.46	22.45	6	—	—		$2'.7$ = 13.5 —	—
— D_0	24.25	24.25	6	—	—		$6'.7$ = 33 —	—

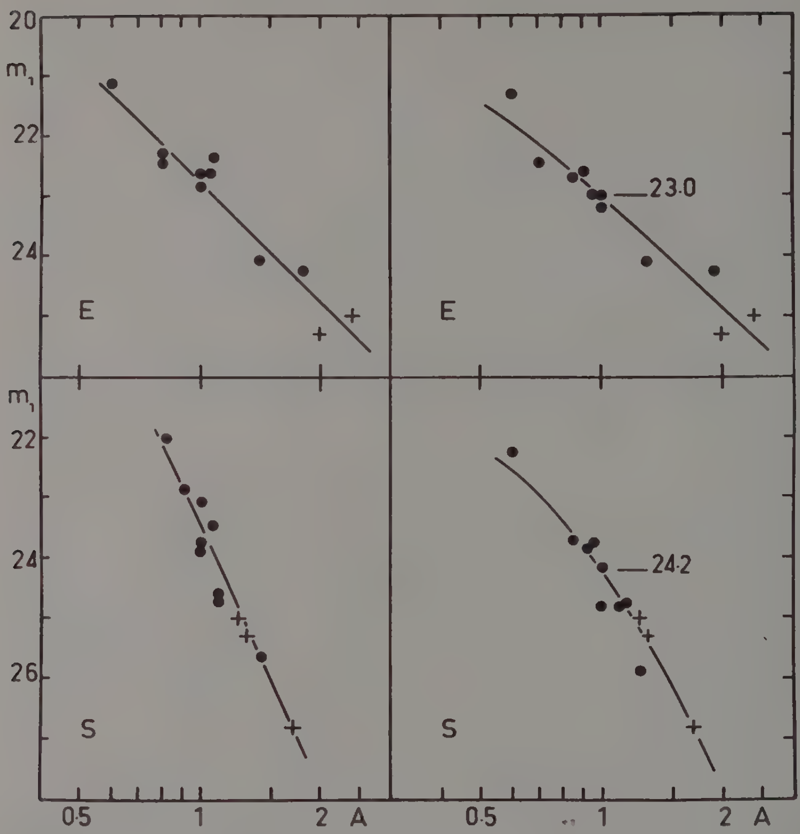

Figure 3. Relation between mean threshold brightness m_1 (mag/sec²) and scale factor A. Empirical values (left), adjusted values (right). $A = 1$ for system Ia.

average for the mean diameter listed in the last part of the table; $m_1' = m_1 + \Delta m_1$ is the value corrected for the variation Δm_1 of threshold brightness as indicated by Figure 2. Both values appear in Figure 3 where m_1 is plotted against the empirical value of A (left) and m_1' against the adjusted value of A (right) as listed in Table III. The crosses are additional data from three series of brightness and microphotometric diameters.

The theoretical relations $m_1(A)$, equivalent to $\log B = f(r)$, corresponding to luminosity laws (1) and (2) are shown by the curves in the right hand part of Figure 3. The average threshold brightness corresponding to $A = 1$ is $m_1 = 23.0$ mag/sec² (E) or 24.2 mag/sec² (So); the latter may also apply to spirals, but a direct verification will be needed. Note that the threshold brightness is not the same for galaxies of different luminosity laws even when measured on the same plates.

Finally, if a general luminosity law characterizes ellipticals, and another characterizes spirals in their outer regions, then a statistical relation should exist between the scale coefficients A relating to each type. A plot of A(E) against A(S) shows that this is indeed the case and the mean empirical relation $A(\mathrm{S}) \simeq \sqrt{A(\mathrm{E})}$ is in good agreement, at least for $A \geqslant 1$, with the theoretical relation derived from the luminosity laws (1) and (2) if one assumes as before that $\alpha = 1$ (E) or 2 (S) for $A = 1$. This relation was also used as a guide to homogenize the values of A in Table III.

4. *Provisional reduction to a standard system.* A provisional reduction of series I a, b, c; II a, b; III a, b, c, d; IV; V; VI a, b, c; X; XI a, b; XII and XIII a, b, c to the corrected Heidelberg system (Ia) was made by means of the *non-adjusted* coefficients A, B of Table III. The corrected Heidelberg system is defined as follows:

(i) the axial ratio R(Ia) is corrected according to Holmberg's precepts (1946) which give the following relations between the photometric ratio

TABLE V. REDUCTION OF MICROMETRIC DIMENSIONS TO STANDARD SYSTEM

NGC 4486, E0–1p, $m_T = 11.0$

Series	Observed D	Observed R	Corrected $\log D$	Corrected $-\log R$
Ia	3′.2	1.00	0.56	0.00
Ic	2′.3	0.61	0.52	0.17
IIa	3′.3	1.00	0.51	0.00
IIb	7′.7	0.90	0.63	0.05
IV	3′.5	1.00	0.59	0.00
X	2′.0	1.00	0.47	0.00
XII	2′.3	1.00	0.50	0.00
Mean			0.54	0.03
Aver. dev.			.05	.05

Corrected $\bar{D} = 3'.5 \pm 0'.2$ (p.e.)

— $\overline{d/D} = 0.93 \pm 0.03$ (p.e.)

NGC 4216, SAB(r:)b, $m_T = 10.8$

Series	Observed D	Observed R	Corrected $\log D$	Corrected $-\log R$
Ia	7′.5	0.14	0.90	0.70
Ib	7′.5	0.12	0.90	0.74
Ic	6′.5	0.08	0.88	0.86
IIb	9′.0	0.20	0.83	0.70
IV	8′.7:	0.14:	0.96:	0.69:
X	7′.0	0.15	0.89	0.68
XIa	6′.0	0.17	0.81	0.64
XII	7′.3	0.15	0.90	0.67
XIIIc	7′.7	0.20	0.91	0.57
Mean			0.89	0.69
Aver. dev.			.03	.05

Corrected $\bar{D} = 7'.7 \pm 0'.2$ (p.e.)

— $\overline{d/D} = 0.20 \pm 0.01$ (p.e.)

b/a and the micrometric ratio β/α as measured by Reinmuth (1926)

b/a	β/α (E)	β/α (S)	b/a	β/α (E)	β/α (S)
1.0	1.0	1.0	0.5	0.397	0.429
0.9	0.873	0.882	0.4	0.289	0.322
0.8	0.750	0.766	0.3	0.192	0.223
0.7	0.630	0.652	0.2	0.113	0.137
0.6	0.512	0.540	0.1	0.049	0.062

A comparison of the values so corrected with photometric determinations for the objects of Table I confirms the validity of Holmberg's corrections;

(ii) the diameter D(Ia) is corrected by $+0'.5$ according to the precepts of section 3 to give equivalent brightness dimensions at $m_1 \simeq 23.0$ mag/sec² (E) or 24.2 (S). In order to reduce to a common threshold brightness, say $m_1 = 23.5$ mag/sec², corrected diameters should be multiplied by about 1.25 (E) or 0.80 (S); reduction factors to $m_1 = 25.0$ mag/sec² would be 2.4 (E) or 1.25 (S), but the application of such large corrections is not advisable.

Mean corrected dimensions were computed for about 1200 Shapley-Ames galaxies listed in two or more series. Table V gives two examples, for an elliptical and a spiral. The complete list of corrected diameters for over 2000 galaxies based on all available series will be published later.

Probable errors computed for all objects whose means are based on at least four values indicate that the individual accuracy of dimensions data varies relatively little from one series to another; it is generally within ±30 per cent of the mean p.e. = 0.047 in $\log D/d$, or ±12 per cent in d/D, independent of type; it is within ±50 per cent of the mean p.e. = 0.061 in $\log D$, or 15 per cent in D, for ellipticals, and of the mean p.e. = 0.040 in $\log D$, or 10 per cent in D, for spirals. The different accuracy of diameter data for spirals and ellipticals is a consequence of the different luminosity laws; the corresponding common value of the p.e. of the threshold brightness m_1 is about 0.4 mag.

Thus, if independent dimension data are available for a bright galaxy from, say, four different series, the mean corrected diameter is known with a p.e. of 5 or 7 per cent (S or E) and the axial ratio with a p.e. of 6 per cent (S, E). Further, if adequate calibration data are available, the surface brightness of the corresponding isophote is known with a p.e. of 0.2 mag. This result encourages the belief that micrometric dimension data so corrected could be used as distance indicators (see section 5).

In preparation for this application some factors affecting the apparent diameters of galaxies must be considered. Figure 4 shows the correlation

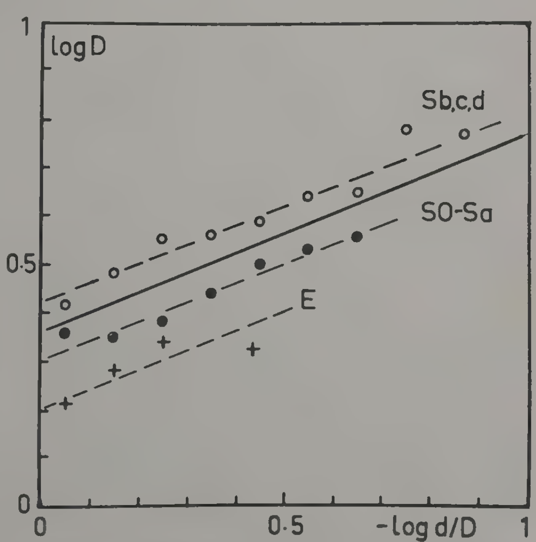

Figure 4. Relation between mean apparent diameter and axial ratio for galaxies of total magnitudes $11.0 < m_T < 12.0$.

between mean apparent diameter D in the corrected Heidelberg system and axial ratio for objects of types E, So–Sa, Sb, c, d and total magnitudes $11.0 < m_T < 12.0$; the relation is essentially the same for all types,

$$\log D = \log D(0) + 0.40 \log D/d \qquad (11)$$

$D(0)$ is the face-on diameter, i.e. the diameter that would be observed for $D/d = 1$ or $\log D/d = 0$. This relation agrees closely with that derived from fewer data in the Mount Stromlo survey (de Vaucouleurs 1956b); on the average a galaxy seen edgewise with an axial ratio $R = d/D = 0.2$ appears with a maximum diameter almost twice its face-on diameter $D(0)$. This, of course, is the cause of the apparent excess of edgewise objects in many statistical surveys (Wyatt & Brown 1955).

Figure 5 shows the correlation between apparent diameter and galactic latitude for objects of axial ratio $R \geqslant 0.5$ and total magnitudes $10.5 < m_T < 12.0$. A variation of the order of $\Delta \log D = +0.15$, or 40 per cent in D, between $b = 20°$ and $b = 70°$ is indicated, and confirms qualitatively earlier results of Reiz (1941) based on the uncorrected Shapley-Ames magnitudes and diameters. The mechanism of this variation has been discussed earlier (de Vaucouleurs 1957a), but the exact form of the relation remains to be determined.

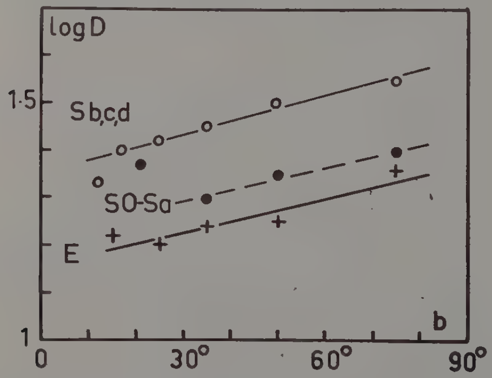

Figure 5. Relation between mean apparent diameter and galactic latitude for galaxies of axial ratio $R \geqslant 0.5$ and total magnitude $10.5 < m_T < 12.0$.

5. *Apparent diameters as distance indicators.* In high galactic latitudes the relation between apparent diameter and total magnitude can be written

$$m_T + 5 \log D(0) = \text{const.}, \qquad (12)$$

where the constant depends only on type and surface brightness.

The scattered distribution of the S–A galaxies as a function of corrected Harvard magnitude m_c and of conventional surface brightness $m_c' = m_c + 2.5 \log Dd$ was illustrated previously (de Vaucouleurs 1956a, 1957a) for the dimension data of the S–A catalogue. A conspicuous reduction of the scatter takes place when the mean corrected diameters are used as shown by Figure 6 giving plots of m_T vs. $D(0)$ for galaxies in the 12 degrees core of the Virgo cluster ($b = +75°$). The constant of relation (12) is 13.0 for ellipticals, 13.5 for So–Sa and 13.8 for Sb, c, d; for the last group the dispersion of individual values hardly exceeds 0.2 mag.

This result does not imply a near uniformity of intrinsic surface brightness among galaxies of a given type; it arises simply from the fact that brightness dimensions are directly correlated and

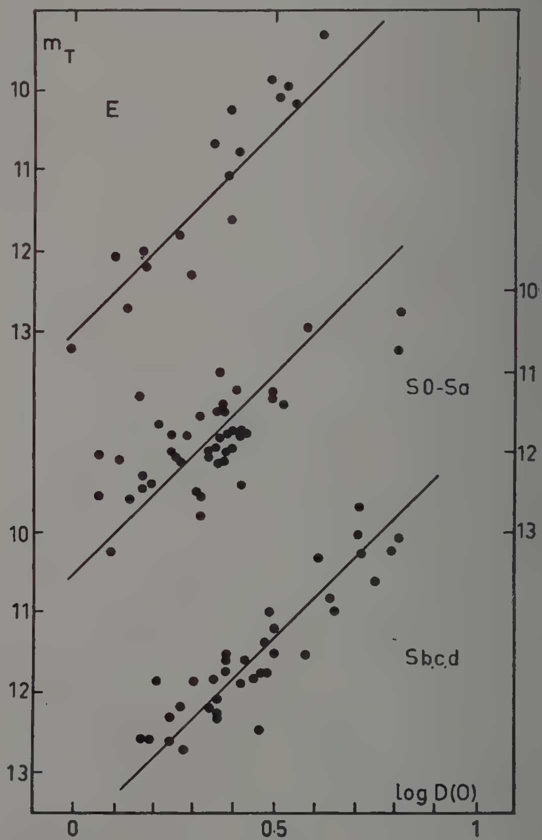

Figure 6. Correlation between total magnitudes and "face-on" diameters of galaxies in the Virgo Cluster.

total magnitudes are inversely correlated with surface brightness, so that the opposite variations of the two quantities balance out to a large extent. It is of some interest that for the brightness dimensions at $m_1 = 25.0$ mag/sec² the constant would be about 14.0 for all galaxy types (excluding low luminosity elliptical and irregular dwarfs); further, since the mean absolute magnitude of galaxies of all types is also a constant (de Vaucouleurs 1958), it follows that the mean brightness dimensions at $m_1 = 25.0$ mag/sec² are also about the same for all types (with the same exclusions).

Because of the close correlation between $D(0)$ and m_T either can be used as a distance indicator. Here the speed and simplicity of determination of micrometric dimensions is an important advantage in comparison with the slow and laborious photometry.

The practical possibility of using reduced diameters as statistical distance indicators was first explored in the Mount Stromlo Survey (de Vaucouleurs 1956b, pp. 29–32); the relative distances of seven clouds or regions in the southern sky so determined were placed on an arbitrary distance scale. The provisional mean diameters now available for nearly all S–A galaxies in both hemispheres provides for a tie with the current distance scale (Sandage 1958) which is essentially based on northern hemisphere objects. Table VI gives the mean reduced diameters and total magnitudes of the largest spirals in some nearby groups whose distance moduli are known; no distinction between apparent and corrected moduli needs to be made at the present degree of accuracy, and with this accuracy the rounded-off mean values $\overline{M}$ (pg) $= -20.0$ and $\overline{D(0)} = 20$ kpc may be adopted. Here again D is the mean diameter in the corrected Heidelberg system and refers to $m_1 \simeq 24.2$ mag/sec². Table VIa gives conversely the distance moduli derived with these values for M31–M33, the Ursa Major group, the Virgo Cluster, and for the compact cloud of spirals in Grus (region VII in the Mount Stromlo Survey). The mean modulus $m - M = 31.2$ places this cloud at a distance of 17.5 Mpc on the current scale, as compared with 7.0 Mpc on the distance scale used in 1956. It follows that the mean distance of region VI (Pavo-Indus) which was used as a unit equal to 10 Mpc in the Mount Stromlo Survey is now 26.5 Mpc and the adopted distances of the other groups are similarly changed; in particular the estimated distance of the Fornax cluster is now about 8 Mpc. More accurate values of the relative distances of these and other groupings of bright galaxies will be derived from the revised dimension data in the final catalogue.

6. *Asymptotic magnitudes*. The initial impetus to secure better diameter data was the need to improve the extrapolation correction $m_r - m_T$ that must be applied to integrated magnitudes m_r to determine the total (or asymptotic) magnitudes m_T.

If the luminosity laws (1) and (2) apply the correction is readily computed. Consider for simplicity circular objects. For ellipticals the total luminosity is, by integration of (1),

$$L_T = 2\pi \int_0^\infty B(r) r dr = 7.215\, \pi B_e r_e^2 \text{ (E)} \quad (13)$$

TABLE VI. MEAN DIAMETERS AND MAGNITUDES OF LARGE SPIRALS

Group	$\overline{D(0)}$	$\overline{m_T}$	$\overline{m_T + 5 \log D(0)}$	$m - M$	$\overline{M}$	$\overline{D}$ (kpc)	Remarks
Local	88′.5:	5.0:	14.5:	24.5	−19.5:	20.5:	(1)
Ursa Maj.	17′.3	8.4	14.4	28.0	−19.6	20	(2)
Virgo	5′.0	10.4	13.9	30.5	−20.1	18	(3)
Grus	3′.5	11.0	13.7	—	—	—	(4)

Remarks:

(1) mean of M31, M33.
(2) mean of M51, M81, M101.
(3) mean of 10 largest (4′ to 6′.5) or brightest (9.7 to 11.0) out of 36 in core of cluster.
(4) mean of 5 largest (3′.0 to 5′.5) or brightest (10.7 to 11.3) out of 19 in region VII of Mt. Stromlo survey.

TABLE VIa. DISTANCE MODULI FOR $\overline{M} = -20.0$ AND $\overline{D} = 20$ KPC

Group	Local	UMa	Virgo	Grus
From $\overline{m_T}$	25.0:	28.4	30.4	31.0
From $\overline{D(0)}$	24.4:	28.0	30.7	31.4
Mean $m - M$	24.7:	28.2	30.6	31.2

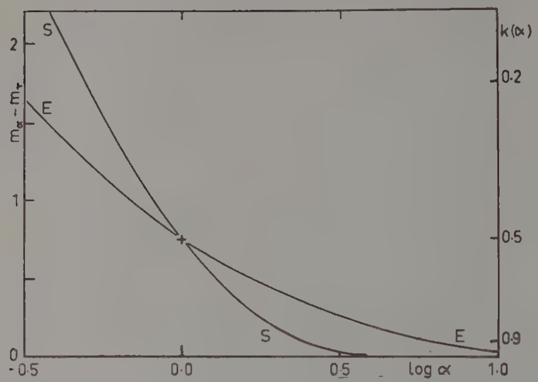

Figure 7. Relative integrated luminosity $m_\alpha - m_T = -2.5 \log k(\alpha)$ as a function of reduced diameter $\alpha = D/D_e$ for the luminosity laws (1) and (2). D_e = effective diameter.

and the fraction L_r/L_T emitted within the isophote of radius r is

$$k(r) = L_r/L_T$$
$$= 1 - e^{-x}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^7}{7!}\right) \quad (14)$$

if $x = 7.668\,\alpha^{1/4}$; by definition $r = r_e$ for $k = \frac{1}{2}$, and $B_e = B(r_e)$.

For spirals the total luminosity of the exponential component is, by integration of (2),

$$\text{(S)} \quad L_T = 3.803\,\pi B_e r_e^2 \quad (15)$$

and the fraction emitted within r is

$$k(r) = 1 - (1 + x)e^{-x} \quad (16)$$

if $x = 1.6785\,\alpha$ and, as above, $k = \frac{1}{2}$ for $\alpha = r/r_e = 1$. Figure 7 gives the variations of $k(r)$ and of

$$m_r - m_T = -2.5 \log k(r) \quad (17)$$

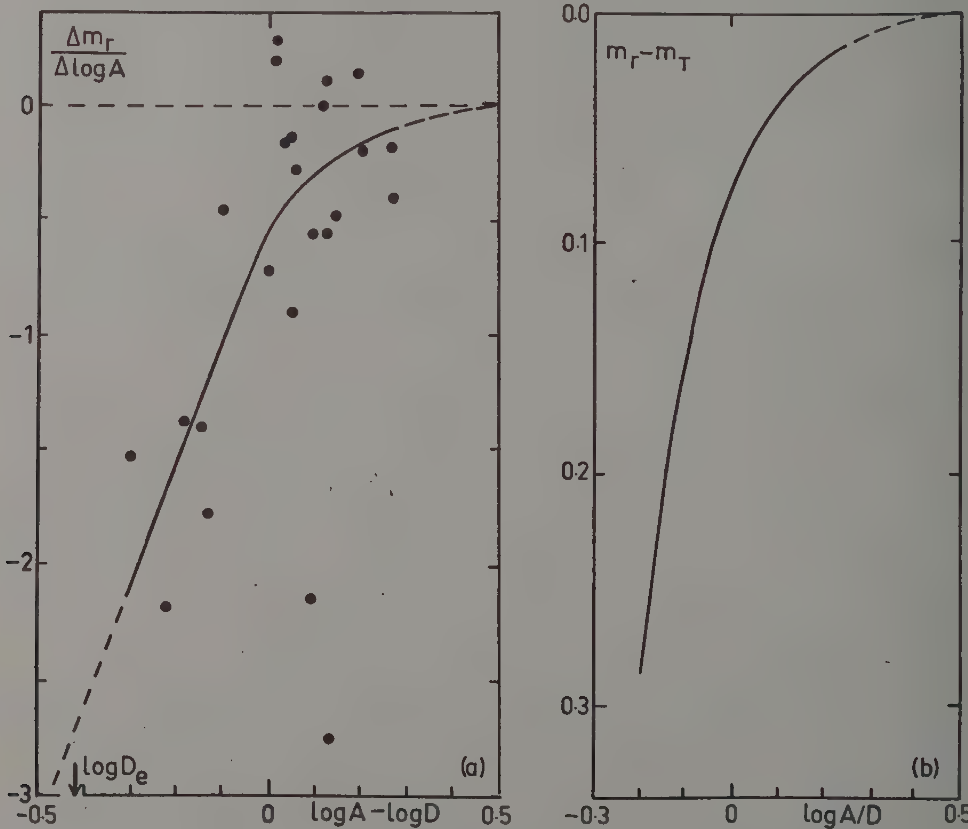

Figure 8. (a) Mean slope $\Delta m_r/\Delta \log A$ of integrated luminosity curve of Sb, Sc spirals as a function of diaphragm aperture $A = 2r$ in the data of Pettit (1954). (b) Empirical incompleteness correction curve $m_r - m_T$ as a function of $\log A/D$. D is the mean micrometric diameter in the corrected Heidelberg system ($\bar{D} \cong 2.6\,\bar{D}_e$).

TABLE VII. NGC 157, SC, $D = 3'.0$

$A = 2r$	2'.3	4'.1	5'.7	Adopted
m_r	11.19	11.01	10.98	
$\log A/D$	−0.115	+0.135	+0.28	
$m_r - m_T$	0.17	0.03	0.01	
$m_{\text{corr.}}$	11.02	10.98	10.97	$\overline{m_T} = 10.99 \pm 0.02$

as a function of α for the two cases. The first, dominating in ellipticals, gives $k(r) > 0.90$, $m_r - m_T < 0.10$ mag. for $\alpha > 5$, $k(r) > 0.97$, $m_r - m_T < 0.03$ mag, for $\alpha > 10$; the second, dominating in late-type spirals, gives $k(r) > 0.90$ for $\alpha > 2.5$, $k(r) > 0.97$ for $\alpha > 3$. Because of the rapid convergence of the luminosity laws (1) and (2) the extrapolation "to infinity" involved in the definition of the asymptotic magnitudes can in fact be limited to more realistic distances.

In order to apply Figure 7 to the statistical correction of measured integrated magnitudes the mean ratio between effective and micrometric dimensions (in the corrected Heidelberg system) is needed. A direct comparison for objects in Table I gives $D \simeq 1.3\, D_e$ for ellipticals, and $D \simeq 1.7\, D_e$ for lenticular galaxies. For spirals an indirect determination was made by means of the photoelectric magnitudes m_r measured by Pettit (1954); Figure 8a shows the variation of $\Delta m_r/\Delta \log A$ as a function of $\log A/D$ for Sb, Sc spirals measured through at least two different apertures $A = 2r$; only objects of axial ratio $R = d/D \geqslant 0.5$ and galactic latitude $|b| \geqslant 45°$ are included.

By integration the empirical relation $m_r - m_T = f(\log A/D)$ shown in Figure 8, b is obtained; it leads to $D \simeq 2.6\, D_e$ and consequently $\log \alpha = \log A/D + 0.42$; with this value the theoretical relation (S) of Figure 7 coincides almost exactly with the empirical relation of Figure 8, b. This result confirms the validity of equation (2), at least in the intermediate and outer regions of spirals.

Table VII gives an example of determination of the asymptotic magnitude of a spiral measured by Pettit (1954). Asymptotic magnitudes so determined will be given in the reference catalogue of bright galaxies now in preparation.

I am much indebted to Mrs. A. de Vaucouleurs for assistance in the numerical work and to the many persons who over the past ten years have made possible the collection of basic data at several observatories. Details of these data will be reported separately.

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MULTIPLE EXPOSURE PHOTOGRAPHIC PHOTOMETRY

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Abstract. A multiple-exposure technique, employing separated photographic and photovisual exposures on the same plate, has been tested for use in galactic cluster photometry. Results, to a given accuracy in $B-V$, are achieved with less work than that required for conventional single exposures.

A special emulsion is recommended, and photometric properties of the Naval Observatory's 40-inch reflector are discussed.

1. *Introduction.* A photometric investigation of a large number of galactic clusters is being carried on jointly by the Lowell and U. S. Naval Observatories. In the course of this program, we intend to measure magnitudes and colors of some 20,000 stars on plates taken with the Naval Observatory's 40-inch reflector. Because of the size of this program, we have sought a method of photographic photometry which will be as efficient as possible. This report deals with a method of multiple exposures on a single plate. Haro has used this method qualitatively with great effectiveness (Haro and Herbig 1955, Haro 1956). Quantitatively, this technique of measuring colors has two advantages over the conventional method of single exposures; first, the number of identification settings in photometering the plates is reduced by a factor of two, and secondly, differential emulsion and processing errors are reduced. Our multiple exposure plates are obtained in the following way: Plates of a given field are taken in pairs. The first exposure is made through a Wratten 47 plus Schott GG13 filter combination on a D sensitized emulsion. The plate is shifted 0.5 mm (15″), and a second exposure is made through a Schott GG11 filter. The second plate of the pair is then taken with the exposures in reverse order. The relative exposure times are adjusted to give approximately equivalent images in blue and yellow for the maximum of the frequency distribution in color for the field. Images are evaluated with an Eichner photometer (Eichner *et al.* 1947; Cameron 1951), and by means of distributed photoelectric standards.

2. *Selection of an emulsion.* In order to appraise the multiple-exposure method, we obtained a series of test plates with various emulsions. Each test plate was multiply exposed to a selected field with two equal exposures separated by 0.5 mm (15″). Pairs of images of a magnitude sequence selected within 25 mm of the plate center were then measured with the Eichner photometer. Each image of each pair was measured twice so that our measuring error could be evaluated. The calibration curves for the various emulsions were all quite similar, giving a scale of about 2 magnitudes per 1000 divisions of the iris scale.

Results of these double-exposure tests can be presented by plotting the mean Eichner reading, $\overline{R}$, against ΔR, the difference in Eichner readings, for the various pairs in a sequence. For a perfect plate with identical exposures, ΔR would be zero in every case. In practice, the $\overline{R}$ vs. ΔR diagrams define a line about which there is some scatter. The scatter of the points may be used as an index of precision, the slope of the line depending on the relative circumstances of the two exposures such as seeing, focus, and guiding. The results of several double exposure tests are summarized in Table I.

TABLE I. RESULTS OF DOUBLE EXPOSURE TESTS

Emulsion	No. of plates	p.e. one measure	p.e. one difference	Exposure for given limit
103a–O	2	$\pm0^m.011$	$\pm0^m.033$	1
IIa–O	2	0.010	0.021	2
III–G	1	0.011	0.021	
IV–D	1	0.012	0.04	
Separation Neg. Type I	2	0.015	0.032	
33	6	0.010	0.016	6
Exp. D	5	0.010	0.016	6

Sample tests of the kind leading to the results shown in Table I are illustrated in Figures 1 and 2. Our initial tests showed that the Eastman 33 plate was superior to normally available spectroscopic emulsions for this work. Accordingly, we asked Mr. William Swann of the Eastman Kodak Company if D-sensitized 33 plates could be supplied for use in multiple exposure photography with blue and yellow filters. This emulsion, designated Experimental D in Table I and Figure 2, was supplied by the courtesy of the Eastman Kodak Company. Additional material of this type, which has been obtained for the cluster program, is designated Experimental CC–D.

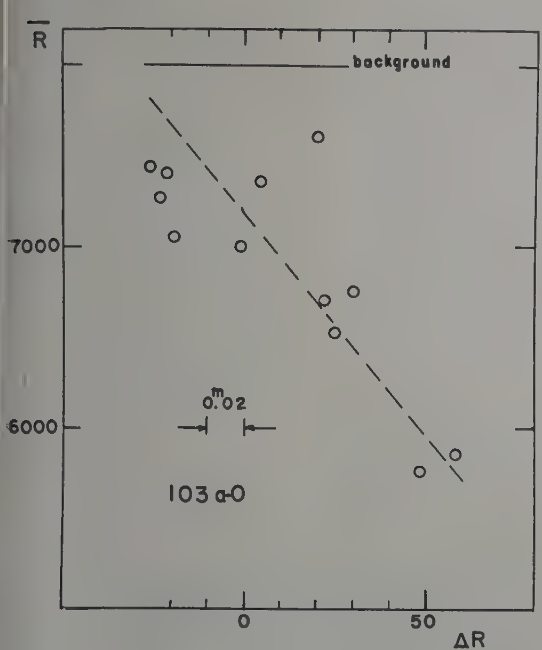

Figure 1. Eichner measures for sample double exposure test of 103a–O emulsion.

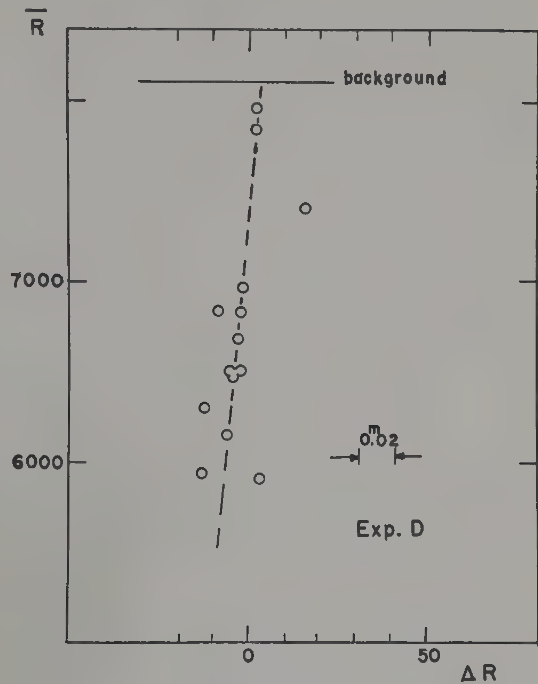

Figure 2. Eichner measures for sample double exposure test of Experimental D emulsion.

Further experiments have shown that the photometric properties of this emulsion are similar to those of the 33 plate.

The last column of Table I gives the relative speeds of four emulsions of interest. It appears that the Experimental emulsion is inconveniently slow. This emulsion, however, responds well to sensitization by baking. We have realized gains of one magnitude for exposures as short as 15 minutes when these plates have been baked for 24 hours at 65° C. This gain in speed is achieved without deleterious effects on fog level or grain size. Response to baking, and consideration of the fact that the ratio of exposure time to measuring time is small, are factors which make the use of the Experimental emulsion practicable.

We found that the index of precision of the Experimental emulsion is not improved by the use of fine grain developer as compared to development in D-19.

Double exposure tests made with either the Wr. 47 + GG13 or GG11 filters used in the program work show an increase in the probable error of the difference between images of a pair, the error introduced by the filters being something of the order of $\pm 0^{m}.010$.

3. *Photometric properties of the Ritchey-Chrétien reflector*. The 40-inch reflector has been described by Hall and Mikesell (1950), and prior to that by Ritchey (1928) and others. This Cassegrain-like telescope has a focal surface concave towards the secondary with a radius of 65 inches, and a scale of approximately 30″/mm. By bending thin plates in a vacuum back, fairly good images can be obtained over a 1°.5 field. However, we prefer to use flat plates for convenience.

When flat plates are used, the field error attributable to curvature of field is large in comparison to the geometrical vignetting of the optical system. The photometric errors produced by this curvature of field were determined by Eichner measures of well distributed photoelectric standards on focus plates. The focus plates were taken in such a way as to provide 10 images covering the range of focus present within a 50 mm (25′) radius from the center of a flat plate. The error to be expected from curvature of field varies according to the relative brightness of the star measured. Figure 3 shows errors determined for images between 1 and 4 magnitudes above the plate limit as a function of the distance from the center of the plate. These values obtain when the best focus is established by knife edge measure at the center of the plate. Note that unsatu-

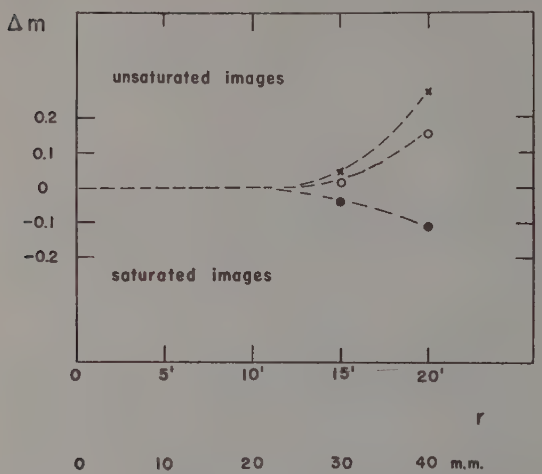

Figure 3. Photometric errors due to curvature of field as a function of distance from the center of the plate. Crosses, open circles, and closed circles represent mean data for images 1, 2, and 4 magnitudes above the plate limit, respectively.

rated images measure fainter at large distances from the plate center when defocused, and saturated images measure brighter. In order to avoid scale errors, the measured field must be kept within a diameter of 20–25′. This field is adequate for the present photometric program.

Astronomical seeing, insofar as aperture, focal length, and site are concerned, may be regarded as a photometric property of the telescope. "Seeing" is measured with the same knife edge apparatus that is used for focusing by the Foucault test. The position of the knife edge is registered by a dial indicator. A *K.E.* cut, i.e., the distance travelled by the knife edge from the position where the first shadows appear on the mirror to the position where the last bright spots disappear, is an index of the performance of the telescope as limited by the seeing and the figure of the mirror. This Foucault shadow measure gives more consistent results than direct measures of the diameters of images. Furthermore, *K.E.* cuts measured in this way give uniform results over large ranges of magnitude or telescope aperture provided that the *K.E.* cut is made slowly enough to integrate all wave front disturbances at the mirror. An effective focal length of 5 meters or more is required for convenient application of this method.

On a given night, the *K.E.* cut will vary with the air mass at which the observation is made. Examples of this variation are shown in Figure 4 for nights of "good" and "bad" seeing. In order

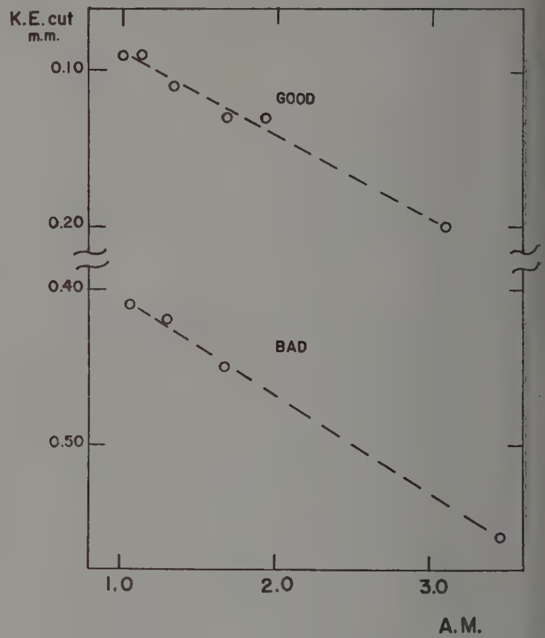

Figure 4. Variation in seeing with change in air mass for two nights with different seeing conditions.

to provide a uniform basis for night-to-night comparisons, *K.E.* cut measures are reduced to the zenith by the following relation:

$$(K.E.)_Z = (K.E.)_{A.M.} - (A.M. - 1)0.05.$$

The *K.E.* cuts are expressed in millimeters and *A.M.* refers to the air mass at which the observation was made.

A frequency distribution of *K.E.* measures for 100 nights distributed more or less at random over a three year period is shown in Figure 5. The three nights illustrated at the tail of the distribution show how miserable conditions can

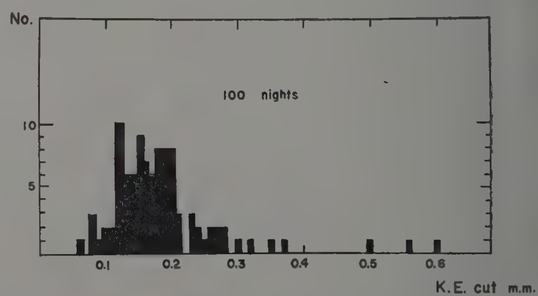

Figure 5. Frequency distribution of seeing measures for a three year period.

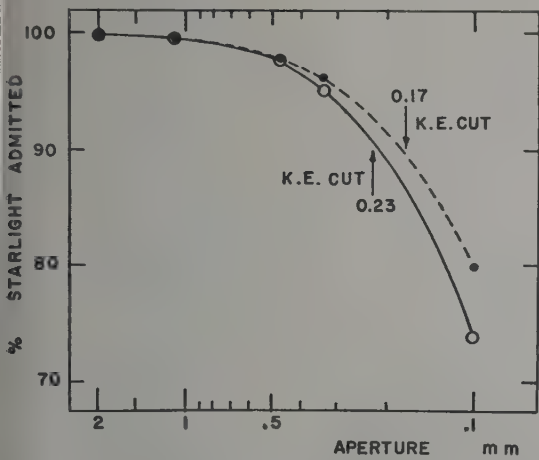

Figure 6. Percentage of starlight admitted through various sized apertures on two different nights.

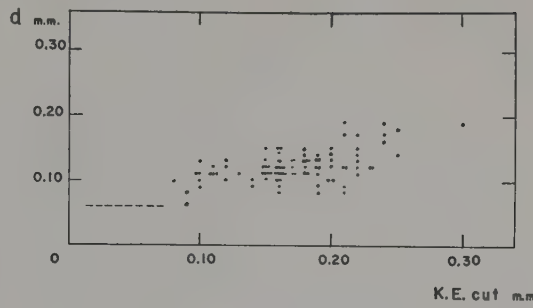

Figure 7. Diameters of photographic images a few tenths of a magnitude above the plate limit as a function of seeing for 100 103a–O chart plates.

get after storm front passages. In order to relate these data to terms more meaningful than units of *K.E.* cut, photoelectric calibrations of these seeing measures have been made in two ways. One approach has been to measure a bright star with a series of different sized apertures. Mean results of series of such measures for two nights are shown in Figure 6. These data show that an aperture diameter equal to the *K.E.* cut on a particular night will admit approximately 90% of a star's light.

Photometer tracings of actual knife edge cuts have provided another means of calibration. From these tracings, the knife edge excursion required to occult the central 68% of the light of an image can be determined. These measures are equivalent to those reported by Meinel (1958) in which occulting bars were used to assess the sizes of images. If the same correction for instrumental scattering applied by Meinel is made, it is found that:

$$\sigma'' = 8.6(K.E.)_{\mathrm{mm}}$$

σ'' is the width of an occulting bar in seconds of arc which will obstruct 68% of the light of an image if correction is made for instrumental scattering.

If we apply this latter calibration conversion to the data illustrated in Figure 5, we find that the average σ'' for our site at Flagstaff is $1''.4$. This same value has been reported by Meinel for 150 nights distributed over one year's time at Kitt Peak for observations of Polaris.

The actual photographic performance of the telescope as related to seeing is illustrated in Figure 7. Diameters of photographic images a few tenths of a magnitude above the plate limit are plotted against the *K.E.* cut index of seeing for 100 103a-O chart plates. Diameter measures were made with a 60-power binocular microscope so as to include all grains attributable to the images. Each point on the diagram represents the mean of several such measures for one plate. The large scatter in the diagram can be explained by errors in focus and guiding, and variations in seeing during a given exposure. Because there is not a one-to-one correspondence between knife-edge cut measures and the limiting image sizes, it is apparent that only the intensity maximum for faint images is effective in producing a photographic image. The "washing-out" of the image intensity profiles as a function of seeing quality results in loss of limiting magnitude for a given exposure because of the relative reduction of central intensity and effective increase in area of the integrated optical images. Knife-edge cut measures are of some use in anticipating the exposure required for a given limiting magnitude.

4. *Preliminary results of multiple exposure photometry.* Preliminary results of photoelectrically calibrated multiple-exposure photometry for 20 galactic clusters are presently available for evaluation. External errors for the photographic work have been estimated by comparing photographic and photoelectric results for about 30 stars in each cluster. Comparisons are made as follows: Separate calibration curves are constructed for the means of the photographic and photovisual measures for pairs of plates. Instrumental magnitudes, b and v, may then be read from these curves. We have so far found that:

$$V = v$$

and

$$B - V = A_1 + A_2(b - v),$$

where A_1 is a function of the photovisual Eichner photometer reading, and

$$A_2 = 0.95(1 + 0.03\,\overline{A.M.}).$$

The term in the parentheses in the latter equation takes into account differential extinction as a function of the mean air mass, $\overline{A.M.}$, and the scale factor, 0.95, has been determined from results for 300 stars. Further details of the reduction procedures will be given when the final results are published.

Probable errors are given in Table II for results from pairs of plates. Arp (1955) has pointed out that IIa-O plates give better results than 103a-O plates. Our data show that a further improvement can be realized by the use of the Experimental emulsion. These data also show that the multiple exposure technique gives better results than an equivalent amount of material on separated single exposures since the errors in $B-V$ are somewhat less than those in V rather than being $\sqrt{2}$ times as great. We also conclude that, for a given amount of plate material, photometric errors cannot be substantially further reduced by improvement in emulsions as the Experimental emulsion error is small in comparison to the total error introduced by other steps in the photometric process.

TABLE II. EXTERNAL PROBABLE ERRORS FOR MULTIPLE EXPOSURE PLATE PAIRS

Emulsion	No. pairs	p.e. (V)	p.e. ($B-V$)
103a–D	2	$\pm 0^{m}\!.036$	$\pm 0^{m}\!.033$
IIa–D	16	0.031	0.026
Exp. CC–D	2	0.027	0.022

5. *Summary*. We have found that a multiple-exposure method of measuring colors photographically gives better results than an equal number of separated exposures. This is presumably brought about by a reduction in the effects of emulsion and processing irregularities. The method is also more efficient in that identification settings with the plate photometer are cut in half, for $B-V$ measures, as compared to measurement of single exposure plates.

We are now using the multiple-exposure method for three-color observations in our galactic cluster program.

Acknowledgments. It is a pleasure to acknowledge that Dr. Harold Johnson has been a continuing source of ideas and constructive work that have led to this report. Braulio Iriarte and Kenneth Hallam have been primarily responsible for the photoelectric work fleetingly referred to in section 4. Dr. Stewart Sharpless has developed a machine method of data handling by which the photographic cluster measures have been processed on the electronic calculator at the U. S. Naval Observatory. A major part of the Eichner measures for the cluster fields so far observed has been made by A. Seeglitz and D. Hart. The general galactic cluster photometry program which furnishes much of the background of this report is being supported at Lowell Observatory by the Office of Naval Research.

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PARALLAX AND MASS RATIO OF 10 URSAE MAJORIS

FROM PHOTOGRAPHS TAKEN WITH THE 24-INCH SPROUL REFRACTOR

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Abstract. Measurement and reduction of photographic plates taken with the Sproul refractor over the interval 1912–1959 yield $+''.0721 \pm ''.0047$ (p.e.) for the relative parallax and $+''.1702 \pm ''.0040$ for the semi-axis major of the photocentric orbit. The adopted value for the absolute parallax, $+''.074 \pm ''.003$, leads to masses .63 $\odot$ and .49 $\odot$ for the primary and secondary components respectively.

Introduction. More than one revolution has been completed since the binary character of 10 Ursae Majoris, $8^h54^m.2$, $+42°11'$ (1900) was discovered by Kuiper (1935). The visual magnitudes of the components given by Eggen (1956) are 4.19 and 6.02; the spectrum has been classified as F5V by Johnson and Morgan (1953). The recently determined orbit by Baize (1955), $P = 22^y.20$, $a = ''.61$, combined with the present Sproul material permits a determination of the masses of the components with substantially greater accuracy than was hitherto possible. The present determination includes the Sproul material used in the earlier solutions for parallax (Miller 1918) and for parallax and mass ratio (van de Kamp 1947).

Material and measures. The procedure described in previous Sproul papers has been followed (van de Kamp and Lippincott 1949). The material includes 608 exposures on 186 plates, taken on 67 nights, resulting in a total weight of 142. The material is summarized in Table I. The regular exposure time was 10 minutes for the interval 1912–1916, 2 minutes for the interval 1939–1945 and $1\frac{1}{4}$ minutes since 1945. The dependences and their yearly changes are given in Table II which also includes the standard frame and position of 10 UMa for 1940.0. The spectra of the reference stars were kindly furnished by Dr. Vyssotsky. The brightness of 10 UMa was reduced to match the average magnitude 10.8 of the reference stars by the use of a rotating sector of .20 per cent opening. The plates were measured on the St. Clair-Kasten long-screw measuring machine in the directions of right-ascension (x) and declination (y) for the approximate equator of the year 2000. The measurements and plate reductions were made by Mrs. John E. Houtman for the material from 1912–1947, and by Mrs. Mary Jackson for the material 1949–1959. Mrs. Jackson also remeasured the plates from the 1940–43 seasons. No significant differences between the two sets of measurements were found; straight means are used with no increase in weight.

Solution for parallax and photocentric semi-axis major. The separation of the components never exceeds $''.7$ or about .04 mm; the measured position of the blended image has been assumed to represent the center of the light intensity or photocenter of the components. The night means are represented by the following equations of conditions:

$$X = c_x + \mu_x t + \pi P_\alpha + \alpha Q_\alpha$$
$$Y = c_y + \mu_y t + \pi P_\delta + \alpha Q_\delta,$$

where the symbols have the usual meaning; t is counted from 1940.00. The orbital factors were computed from Baize's elements (1955):

$$P = 22^y.20 \qquad i = 134°.8$$
$$e = 0.17 \qquad \omega = 211°.8$$
$$T = 1949.97 \qquad \Omega = 26°.5.$$

Separate solutions in right ascension and declination yield the following results:

	mm		p.e.	Weight
c_x	+3.33333			
c_y	+6.57652			
μ_x	−0.022420	$= -''.4231$	$\pm''.0003$	21972
μ_y	−0.012171	$= -.2297$	±.0002	21060
π_x	+0.003833	$= +.0723$	±.0057	66.05
π_y	+0.003906	$= +.0737$	±.0125	7.47
α_x	+0.008558	$= +.1615$	±.0069	44.74
α_y	+0.009394	$= +.1773$	±.0046	55.38
p.e. 1_x	±0.0025	= ±.046		
p.e. 1_y	± .0018	= ±.034		

A combined solution for π and α gives the following results:

	mm		p.e.	Weight
c_x	+3.33336			
c_y	+6.57640			
μ_x	− .022416	$= -''.42300$	$\pm''.00027$	22330
μ_y	− .012166	$= -.22958$	±.00027	22368
π	+ .003820	$= +.0721$	±.0047	73.65
α	+ .009022	$= +.1702$	±.0040	100.89
p.e. 1	± .0021	= ±.040		

TABLE I. OBSERVING DATA, MEASURED POSITIONS AND RESIDUALS

Epoch	Date	H.A. min.	No. of pl.	No. of exp.	Obs.	$P\alpha$	$P\delta$	$Q\alpha$	$Q\delta$	X unit .0001 mm	Y unit .0001 mm	v_x unit .0001 mm	v_y unit .0001 mm	p
1912.905	Nov. 26	−45	1	3	B	+.87	−.07	−.64	−.87	+3 9399	+6 8977	+16	−3	1
.911	Nov. 28	−21	1	3	B	+.86	−.05	−.64	−.87	9397	8961	+16	−18	1
.941	Dec. 9	−15	1	3	M	+.76	+.04	−.64	−.88	9391	9006	+21	+28	1
.949	Dec. 12	−39	1	3	B	+.73	+.07	−.64	−.88	9382	8955	+15	−23	1
.952	Dec. 13	−12	1	3	M	+.72	+.08	−.64	−.88	9365	9005	−1	+27	1
.957	Dec. 15	−39	1	3	B	+.70	+.09	−.64	−.88	+3 9354	+6 9003	−10	+25	1
13.222	Mar. 22	+15	1	3	M	−.74	+.46	−.61	−.92	9250	8919	−3	−37	1
.965	Dec. 18	+3	1	3	P	+.66	+.12	−.51	−1.01	9106	8872	−44	+27	1
15.163	Mar. 1	−10	1	2	P	−.46	+.49	−.32	−1.10	8847	8728	−8	+23	1
.248	Apr. 1	−18	1	2	M	−.83	+.43	−.31	−1.10	8809	8651	−12	−41	1
.256	Apr. 4	−13	1	2	P	−.85	+.42	−.31	−1.10	+3 8792	+6 8700	−26	+10	1
.267	Apr. 8	+4	1	2	M	−.88	+.40	−.31	−1.10	8842	8656	+27	−33	1
.996	Dec. 30	+3	1	2	P	+.53	+.20	−.18	−1.10	8726	8612	+9	+20	1
16.007	Jan. 3	+5	1	2	S	+.46	+.24	−.18	−1.10	8722	8616	+10	+24	1
.187	Mar. 9	−1	1	3	S	−.58	+.49	−.14	−1.10	8644	8587	+8	+7	1
.193	Mar. 11	+7	1	3	Ma	−.61	+.48	−.14	−1.10	+3 8631	+6 8596	−3	+16	1
39.134	Feb. 18	−8	2	4	P	−.29	+.47	−.01	−1.07	3542	5809	+25	+18	2
.270	Apr. 9	+2	1	2	D	−.89	+.40	+.02	−1.06	3371	5740	−94	−32	1
.909	Nov. 29	−29	4	14	T	+.86	−.06	+.14	−1.00	3412	5684	+13	+2	3
40.895	Nov. 23	−20	2	6	P	+.90	−.10	+.31	−0.88	3187	5558	−7	−13	2
41.108	Feb. 8	0	4	13	T	−.13	+.45	+.35	−.84	+3 3106	+6 5552	−5	−18	3
.900	Nov. 25	−20	4	12	T	+.88	−.09	+.47	−.70	2958	5459	−26	−7	3
43.273	Apr. 10	−6	4	12	D	−.89	+.40	+.65	−.40	2616	5341	−9	−4	3
.276	Apr. 11	−18	2	6	D	−.90	+.39	+.65	−.40	2627	5337	+4	−7	2
.898	Nov. 25	−18	2	4	D	+.89	−.09	+.70	−.25	2560	5257	+3	−6	2
44.245	Mar. 30	−12	4	10	D	−.82	+.44	+.71	−.16	+3 2417	+6 5236	+3	−13	3
.942	Dec. 10	+4	4	8	D	+.76	+.04	+.75	+.02	2320	5160	−3	−6	3
45.236	Mar. 27	+3	4	9	D	−.78	+.45	+.75	+.10	2200	5158	+3	+5	3
.938	Dec. 9	+6	4	15	D	+.78	+.03	+.74	+.28	2092	5049	−7	−19	3
46.965	Dec. 19	+20	2	6	Y	+.67	+.11	+.65	+.52	1810	4971	−48	+3	2
47.175	Mar. 5	−14	2	6	N	−.51	+.49	+.63	+.56	+3 1748	+6 4935	−13	−24	2
.194	Mar. 12	+2	2	4	S	−.61	+.48	+.62	+.56	1738	4975	−16	+17	2
.210	Mar. 18	−14	4	15	D	−.68	+.48	+.62	+.57	1727	4945	−20	−11	3
.246	Mar. 31	+4	2	8	Bi	−.82	+.44	+.62	+.57	1766	4950	+33	0	2
.262	Apr. 6	−17	4	12	D	−.86	+.41	+.61	+.58	1707	4922	−21	−27	3
.972	Dec. 22	−28	4	14	Ro	+.64	+.13	+.50	+.69	+3 1656	+6 4829	+40	−32	3
48.199	Mar. 13	−17	4	16	Ro	−.63	+.48	+.46	+.72	1506	4868	−7	+19	3
.928	Dec. 5	−24	4	15	Ho	+.81	00	+.30	+.78	1433	4750	+43	+2	3
49.859	Nov. 10	−27	4	16	Ro	+.94	−.21	+.07	+.78	1154	4650	−12	+23	3
50.148	Feb. 23	−6	4	16	Ru	−.37	+.48	−.01	+.76	1091	4644	+48	+28	3
.965	Dec. 19	−22	4	16	Da	+.67	+.11	−.22	+.65	+3 0862	+6 4501	−21	+8	3
52.196	Mar. 12	−28	4	16	Fr	−.62	+.48	−.49	+.39	0536	4355	+5	+22	3
53.122	Feb. 13	0	4	16	Fr	−.21	+.46	−.64	+.14	0342	4191	+15	−7	3
.856	Nov. 9	−6	2	5	Wy	+.95	−.22	−.72	−.07	0216	4107	+18	+44	1
.916	Dec. 1	−8	4	16	Fr	+.84	−.04	−.72	−.08	0193	4069	+13	+7	3
54.209	Mar. 17	−7	4	16	Wy	−.67	+.48	−.74	−.17	+3 0060	+6 4054	+4	+16	3
.211	Mar. 18	−1	4	11	fl	−.68	+.48	−.74	−.17	0079	4051	+23	+13	3
55.121	Feb. 13	−1	2	7	Ra	−.21	+.46	−.77	−.41	2 9854	3887	−13	−18	2
.140	Feb. 20	−2	4	16	Ra	−.32	+.48	−.77	−.42	9928	3944	+70	+42	3
.224	Mar. 23	−10	4	10	Ra	−.74	+.46	−.77	−.44	9820	3903	−4	+13	2
.243	Mar. 30	−24	4	16	Wy	−.81	+.44	−.77	−.44	+2 9795	+6 3897	−22	+10	3
.274	Apr. 10	−6	4	11	Ra	−.89	+.40	−.77	−.45	9812	3887	+5	+6	2
.872	Nov. 15	−12	2	8	bi	+.93	−.18	−.75	−.60	9759	3775	+15	+3	2
56.014	Jan. 6	−22	4	9	bi	+.43	+.25	−.75	−.62	9678	3758	−16	−12	2
.016	Jan. 7	−7	2	5	Fr	+.42	+.26	−.75	−.62	9666	3781	−25	+12	1

TABLE I (*continued*)

Epoch	Date	H.A. min.	No. of pl.	exp.	Obs.	$P\alpha$	$P\delta$	$Q\alpha$	$Q\delta$	X unit .0001 mm	Y unit .0001 mm	v_x unit .0001 mm	v_y unit .0001 mm	p
.251	Apr. 1	−11	4	14	Fr	−.83	+.43	−.73	−.68	+2 9598	+6 3743	+5	+1	3
.262	Apr. 5	−7	4	14	Po	−.86	+.42	−.73	−.68	9600	3743	+10	+3	3
.874	Nov. 15	−28	4	16	Po	+.93	−.17	−.69	−.80	9494	3619	−31	−14	3
57.133	Feb. 17	−30	4	14	Jo	−.28	+.47	−.66	−.84	9391	3628	−33	+6	3
.166	Mar. 1	−10	2	8	Br	−.46	+.49	−.66	−.85	9447	3593	+39	−24	1
58.182	Mar. 7	+8	4	14	Br	−.54	+.49	−.54	−.99	+2 9220	+6 3465	+31	−16	2
.222	Mar. 22	−28	2	5	Sc	−.73	+.47	−.53	−1.00	9183	3491	+9	+16	1
.951	Dec. 14	−8	1	4	Jo	+.73	+.07	−.42	−1.06	9070	3372	−6	+7	1
59.140	Feb. 20	−30	4	16	Wd	−.32	+.48	−.39	−1.08	8978	3331	−18	−25	3
.142	Feb. 21	−35	4	16	Jo	−.33	+.48	−.39	−1.08	8983	3342	−12	−14	3
.145	Feb. 22	−35	4	16	Jo	−.35	+.48	−.39	−1.08	+2 8988	+6 3351	−6	−5	3
.156	Feb. 26	0	4	8	Wy	−.41	+.48	−.39	−1.08	8948	3345	−40	−9	2

B = Samuel G. Barton
Bi = Leendert Binnendijk
bi = Edwin V. Bishop
Br = Sheila V. Brown
Da = John J. Davis, Jr.
D = Roy W. Delaplaine
fl = Edith Flather
Fr = Laurence W. Fredrick
Ho = Louis N. Howard
Jo = F. Jerrold Josties
Ma = Walter A. Matos
M = John A. Miller
N = Edward P. Neuburg
P = John H. Pitman
Po = William Poole, Jr.
Ra = F. Duane Ray
Ro = Karl Hans Roth
Ru = Joseph D. Rutledge
Sc = T. Paul Schultz
S = Hannah B. Steele
s = Walter A. Strauss
T = Armstrong Thomas
Wd = H. John Wood
Wy = Arne A. Wyller
Y = George B. Yntema

TABLE II. REFERENCE STARS

No.	m_{pv}	Spectrum	Diameter mm	x_s mm	y_s mm	Dep. 1940	ΔD/yr
1	10.8	F5	.137	−58.55	+12.16	.363	+.00009
2	11.1	G0	.117	+3.30	−32.40	.203	+.00025
3	11.0	K0	.125	+55.25	+20.24	.434	−.00034
10 UMa	(10.8)	F5V	.137	+3.34	+6.57		

Table I gives the residuals from the combined solution for each night.

Discussion. Table III gives the mean epoch for the normal places of residuals in x and in y, together with the total weight and number of nights. These residuals in seconds of arc are plotted with respect to the photocentric orbit in the central portion of Figure 1. In the outer part of the figure, the visual observations are plotted with Baize's orbit for comparison. Between 1940 and 1945 and again between 1952 and 1955, the photographic positions fall behind the orbital ephemeris. The visual material is stronger than the photographic material and shows no similar anomalies. Visual positions since Baize's orbit determination have been added. Although all the observations near apastron tend to fall outside the orbit, it is probably premature to revise the orbit at this time.

The present value for the relative parallax $+''0721 \pm ''0047$, supersedes the earlier Sproul determination based on part of the present investigation. From the average magnitude 10.8 and spectrum G2 of the reference stars, at galactic latitude +43°, we find $+''004$ for the mean parallax of the reference system (Vyssotsky and Williams 1948), (Binnendijk 1943). This leads to the value $+''076$ for the Sproul absolute parallax of 10 UMa.

TABLE III. NORMAL PLACES OF RESIDUALS

Epoch	Σp	Σn	v_x unit .0001 mm	v_y unit .0001 mm
1912.98	7	7	+8	0
15.48	9	9	+4	+6
40.22	11	5	−3	−6
43.28	13	5	−6	−8
45.37	9	3	−2	−7
47.18	14	6	−15	−9
48.37	9	3	+25	−4
50.32	9	3	−5	+20
52.66	6	2	+10	+8
54.09	10	4	+14	+15
55.20	12	5	+10	+13
56.12	11	5	+2	0
57.38	10	5	−8	−6
59.13	12	5	−16	−12

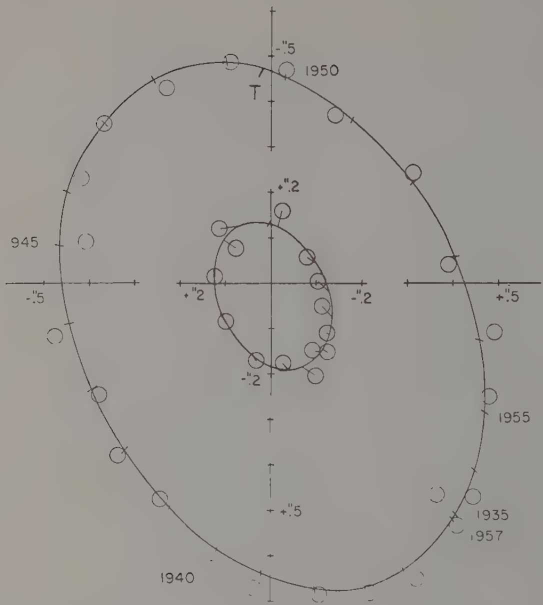

Figure 1. Visual and photographic observations of 10 Ursae Majoris.

The outer part shows the visual observations (B with respect to A at the origin) from 1935 to 1957 grouped in yearly means, and the relative orbit (Baize 1955). The central part shows normal places of the Sproul photographic positions, over the interval 1912–1959 (Table III), after eliminating parallax and proper motion; the center of mass of the system is at the origin. They define the scale of the photocentric orbit whose elements are those of Baize's visual orbit except for scale and a rotation of 180° in the plane of the sky. The central portion of the figure has been rotated 180° with respect to the outer portion to permit a direct comparison of the visual and photographic observations and to illustrate graphically the comparative sizes, $\alpha/a = B - \beta$, of the similar photocentric and relative orbits.

Other trigonometric parallax determinations adjusted for the Yale precepts (Jenkins 1952) are

Allegheny	$+''.069 \pm ''.008$ (p.e.)
McCormick	$+.074 \pm .011$ (p.e.).

The weighted mean value combined with the current Sproul value (unadjusted for Yale precepts) yields

$$\pi_{abs} = +''.074 \pm ''.003 \text{ (p.e.)}.$$

The value of α from the combined solution leads to

$$\frac{\alpha}{a} = B - \beta = +.278 \pm .007 \text{ (p.e.)}$$

assuming no error in the value for a. The value for $\Delta m = 1.83$ given by Eggen (1956) leads to $\beta = .156$.

Adopting the above P, a and π_{abs}, we find:

$$a = 8.2 \pm .3 \text{ a.u.}$$

$$M_A + M_B = 1.12 \pm .14\odot$$

the p.e. arising from the error in a (a.u.) alone.

The individual masses are

$$M_A = .63\odot, \qquad M_B = .49\odot$$

The present investigation places the masses for 10 UMa A and B in quality 2 group of van de Kamp's compilation of masses (1954). The adopted parallax yields absolute visual magnitudes 3.54 and 5.37 for the individual components. Using the bolometric corrections adopted by Eggen (1956), we find absolute bolometric magnitudes 3.49 and 5.31 for the A and B components respectively. Both stars clearly lie above the main sequence mass-luminosity curve. This departure from expected positions appears greater than their positions on the H-R diagram or color-magnitude diagram compared to main sequence stars.

Permissible changes in parallax, semi-axis major and period can hardly explain this deviation. Eggen's photometric parallax (1956) of $''.062$ would bring the masses in agreement with the mass-luminosity relation of main sequence stars; however, this seems an unlikely value in view of the three trigonometric parallax determinations and the small p.e. of their mean. The period cannot be changed sensibly, while an increase of the order $+''.15$ in a, would imply abnormally large systematic errors in the visual observations. The major axis lies close to the line of nodes, which strengthens the value of a derived by Baize. The undermassive quality of both components remains unaccounted for. Further observations of the parallax and orbital motion of this close pair are desired.

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PARALLAX AND MASS–RATIO OF FURUHJELM 46

FROM PHOTOGRAPHS TAKEN WITH THE 24-INCH SPROUL REFRACTOR

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Abstract. Measurement and reduction of photographic plates taken with the Sproul refractor over the interval 1938–1958 yield $+''.1582 \pm ''.0041$ (p.e.) for the relative parallax and $+''.0537 \pm ''.0039$ for the semi-axis major of the photocentric orbit. The adopted value for the absolute parallax, $+''.160 \pm ''.003$, leads to masses $.26\odot$ and $.25\odot$ for the primary and secondary components respectively.

Introduction. Furuhjelm 46 = BD $+45°2505$ = HD 155876, $17^h9^m.2$, $+45°50'$ (1900) is a visual binary with period 13.02 years (Van Biesbroeck 1959) and semi-axis major $''.71$ (Baize 1952). The combined magnitude is 9.38, the magnitude difference $\Delta m = 0.37$ and spectrum of the brighter component is dM4 (Eggen 1956). The radial velocity of the system is -18 km/sec (Wilson 1953). Recent adjustments of the orbital elements combined with a 50 per cent increase in photographic material justify a new determination of the parallax and mass ratio. The current investigation includes the material used earlier and supersedes the former results (Binnendijk 1951).

Material and measures. The procedure described in previous Sproul papers has been followed (van de Kamp and Lippincott 1949). The material includes 860 exposures on 230 plates, taken on 65 nights, resulting in a total weight of 171. The material is summarized in Table I. The regular exposure time for one image was about 7 minutes in 1938 and 1939, $1\frac{1}{2}$ minutes between 1940–1945 and $\frac{1}{2}$ minute for 1946 on. The dependences and their yearly changes are given in Table II which also includes the standard frame and position of Furuhjelm 46 for 1940.0. The spectra for the reference stars were kindly furnished by Dr. Vyssotsky. The image of Furuhjelm 46 was reduced to magnitude 10.0 by the use of a rotating sector of 58 per cent opening, thus providing close magnitude compensation with the three reference stars. The plates from 1938 to 1950 were measured by the earlier investigator (Binnendijk 1951). The 1951–1958 plates were measured and reduced by Mrs. Edward Allen who also remeasured the plates taken between 1941 April 8 and 1943 July 19. The average agreement between the two measurements is well within a micron; the average for each night was adopted for further use without increase in weight. All plates were measured on the Gaertner Machine.

Solution for parallax and photocentric semi-axis major. The maximum separation of the components is $1''.1$ or .06 mm; the images appear elongated under good seeing near the times of maximum separation 1944–1946 and 1957–1958. However, it has been assumed that the measured position of the blended image always represents the center of light intensity or the photocenter of the two components. The night means are represented by the following equation of conditions:

$$\xi - (\gamma_x) = c_x + \mu_x t + \pi P_\alpha + \alpha Q_\alpha$$
$$\eta = c_y + \mu_y t + \pi P_\delta + \alpha Q_\delta,$$

where the symbols have their usual meaning, t is counted for 1940.000. The values prior to 1941.822 were corrected for a color equation γ_x, amounting to $+.0016$ mm due to the adjustment of the components of the objective on that date; the color equation after 1949.05 amounts to $+.0004$ mm (Lippincott 1957).

The orbital factors were computed from Baize's orbit determination (Baize 1958) except for changes in period and periastron passage recommended by Van Biesbroeck (1959) in view of subsequent observations:

$$P = 13^y.02 \qquad i = \pm 153°.9$$
$$T = 1952.19 \qquad \omega = 113°.2$$
$$e = .73 \qquad \Omega = 175°.0$$
$$a = ''.71$$

Separate solutions in right ascension and declination yield the following results:

		p.e.	Weight
$c_x = +8.59639$ mm			
$c_y = +1.86911$			
$\mu_x = +0.015718$	$= +0''.2966$	$\pm 0''.0005$	3981
$\mu_y = -0.087320$	$= -1.6477$	± 0.0006	3955
$\pi_x = +0.008312$	$= +0.1568$	± 0.0041	63.91
$\pi_y = +0.008924$	$= +0.1684$	± 0.0125	9.37
$\alpha_x = +0.002585$	$= +0.0488$	± 0.0050	41.92
$\alpha_y = +0.003161$	$= +0.0597$	± 0.0062	38.65
p.e. $1_x = \pm 0.00174$	$= +0.0328$		
p.e. $1_y = \pm 0.00205$	$= +0.0387$		

TABLE I. OBSERVING DATA, MEASURED POSITIONS AND RESIDUALS

Epoch	Date	H.A. min.	No. of pl.	exp.	Obs.	$P\alpha$	$P\delta$	$Q\alpha$	$Q\delta$	X unit .0001 mm	Y unit .0001 mm	v_x unit .0001 mm	v_y unit .0001 mm	p
1938.239	Mar. 29	−5	2	4	P	+.935	+.289	−.05	+.67	+8 5829	+2 0194	+52	−83	1
.332	May 2	−16	2	4	K	+.603	+.733	−.08	+.62	5797	2 0225	+34	−7	1
39.211	Mar. 19	−17	2	4	D	+.978	+.129	−.19	−.17	5941	1 9380	+12	−11	1
.501	July 2	−12	2	5	D	−.383	+.888	+.04	−.37	5867	9231	−8	+78	1
40.500	July 1	−2	4	11	T	−.379	+.890	+.70	−.35	6047	8329	+3	+5	3
.514	July 6	−7	4	14	T	−.455	+.860	+.70	−.34	+8 6039	+1 8317	0	+7	3
41.267	Apr. 8	−10	2	7	P	+.865	+.440	.99	−.18	6295	7629	+19	+5	2
.338	May 4	+5	2	7	Sl	+.572	+.755	1.02	−.17	6267	7581	+3	−6	2
.538	July 15	−9	4	14	T	−.581	+.793	1.08	−.12	6225	7387	+25	−30	3
42.289	Apr. 16	−15	4	12	k	+.793	+.546	1.24	+.07	6429	6739	+6	−6	3
.409	May 30	0	4	16	D	+.175	+.925	+1.26	+.10	+8 6389	+1 6673	−1	−1	3
43.332	May 2	−30	4	16	D	+.607	+.730	1.38	.32	6571	5863	−4	+6	3
.547	July 19	+4	4	16	D	−.628	+.761	1.39	.37	6500	5668	−5	−5	3
44.249	Apr. 1	−29	4	15	D	+.915	+.341	1.42	.53	6736	5037	−10	+7	3
.364	May 13	−7	2	8	D	+.439	+.834	1.42	.55	6722	4987	−2	+16	2
.514	July 6	−38	4	15	D	−.454	+.860	+1.42	+.58	+8 6671	+1 4825	−2	−19	3
.525	July 10	−15	4	16	D	−.514	+.832	1.42	.58	6691	4807	+22	−23	3
.607	Aug. 9	+1	2	8	D	−.867	+.511	1.42	.60	6643	4720	−9	−13	2
45.232	Mar. 26	−45	4	13	D	+.950	+.246	1.39	.72	6886	4201	−17	+32	3
.341	May 5	−34	2	5	D	+.559	+.764	1.39	.74	6872	4100	−15	−17	2
45.475	June 22	−3	4	15	D	−.230	+.929	+1.38	+.77	+8 6839	+1 4056	−2	+41	3
.505	July 3	−27	4	15	D	−.405	+.880	1.38	.77	6805	3986	−26	+1	3
.513	July 6	−15	4	13	D	−.451	+.862	1.38	.78	6832	3982	+4	+5	3
46.193	Mar. 12	−34	2	7	m	+.989	+.019	1.31	.88	7052	3365	−2	+50	2
.343	May 6	−1	4	16	T	+.551	+.770	1.30	.91	7006	3263	−35	+15	3
.474	June 22	−5	3	12	Ke	−.226	+.930	+1.28	+.93	+8 6991	+1 3119	−4	−29	3
.523	July 10	−15	4	16	D	−.506	+.835	1.27	.93	6978	3095	−2	−1	3
.548	July 19	−2	4	16	Ke	−.631	+.759	1.27	.93	6988	3067	+15	−2	3
.592	Aug. 4	−15	4	16	D	−.816	+.583	1.26	.94	6972	3015	+7	−1	3
.594	Aug. 5	+8	2	8	Ke	−.826	+.570	1.26	.94	6961	2994	−3	−19	2
47.209	Mar. 18	−20	4	15	Y	+.980	+.111	+1.17	+1.01	+8 7211	+1 2445	+2	+5	3
.356	May 11	−30	4	16	D	+.480	+.812	1.14	1.03	7172	2388	−18	+17	3
48.230	Mar. 25	−17	4	16	Ro	+.954	+.234	0.96	1.09	7354	1565	−8	+5	3
.276	Apr. 11	−9	4	16	Ro	+.837	+.485	.95	1.09	7365	1549	+7	+7	2
.462	June 17	−36	4	16	Bi	−.151	+.941	.90	1.10	7281	1420	−23	+2	3
.508	July 4	−50	4	16	Ro	−.424	+.873	+.89	+1.10	+8 7271	+1 1362	−17	−9	3
.519	July 8	−33	4	16	Ro	−.484	+.847	.89	1.10	7287	1374	+2	+14	3
49.218	Mar. 21	−33	4	16	Bi	+.970	+.167	.70	1.09	7520	0698	+5	+6	3
.270	Apr. 9	−12	4	15	Dt	+.856	+.454	.68	1.09	7506	0667	−7	−4	3
.300	Apr. 20	−40	4	15	Ro	+.748	+.600	.67	1.09	7517	0649	+9	−8	3
.328	Apr. 30	−45	4	16	Ho	+.627	+.714	+.66	+1.09	+8 7508	+1 0642	+6	−1	3
.366	May 14	−35	1	4	Dt	+.428	+.840	.65	1.08	7508	1 0633	+16	+14	1
50.267	Apr. 8	+13	4	16	Bi	+.867	+.436	.35	0.98	7669	9788	+7	−8	3
.286	Apr. 15	+13	4	15	Bi	+.803	+.533	+.34	.98	7655	9801	−3	+13	3
51.367	May 15	−4	4	16	Da	+.420	+.843	−.09	.61	7793	8844	+8	−15	3
.550	July 20	−14	4	16	Da	−.641	+.753	−.16	+.50	+8 7716	+ 8680	−6	−9	3
.564	July 25	−18	4	16	Da	−.703	+.702	−.17	+.49	7713	8659	−6	−13	3
.577	July 30	−24	4	16	Da	−.761	+.646	−.17	+.48	7705	8659	−11	+3	3
52.271	Apr. 9	+3	4	16	Fr	+.854	+.457	−.16	−.21	7972	8009	+11	−5	3
.451	June 13	−6	4	14	Fr	−.084	+.946	−.02	−.33	7911	7888	−4	−6	3
53.278	Apr. 12	−8	2	6	Wo	+.830	+.495	+.57	−.39	+8 8133	+ 7108	−5	−25	2
.461	June 17	−11	4	16	Fr	−.146	+.941	.66	−.36	8067	7035	−21	+24	3
.467	June 19	0	4	16	Fr	−.180	+.937	.67	−.36	8083	7002	−3	−4	3
54.327	Apr. 30	−4	4	14	Fr	+.630	+.712	1.01	−.17	8347	6253	+48	+12	3
.392	May 24	−2	4	15	fl	+.275	+.899	1.02	−.16	8267	6206	−12	+5	3

TABLE I (*continued*)

Epoch	Date	H.A. min.	No. of pl.	exp.	Obs.	$P\alpha$	$P\delta$	$Q\alpha$	$Q\delta$	X unit .0001 mm	Y unit .0001 mm	v_x unit .0001 mm	v_y unit .0001 mm	p
55.332	May 2	−8	4	16	Fr	+.607	+.730	+1.25	+.07	+8 8452	+5373	−10	+1	3
.370	May 16	−8	4	16	Fr	+.405	+.850	1.25	.08	8445	5350	−6	0	3
.427	June 6	−11	4	16	Fr	+.064	+.942	1.26	.10	8452	5308	+21	0	3
.493	June 29	−8	4	15	Wy	−.334	+.904	1.27	.11	8411	5218	+2	−29	3
.526	July 11	−4	2	7	Ko	−.517	+.830	1.28	.12	8437	5217	+39	+5	1
.534	July 14	−22	4	13	Fr	−.559	+.806	+1.28	+.12	+8 8412	+5195	+16	−8	2
56.222	Mar. 22	−9	4	16	Po	+.966	+.186	1.36	.29	8604	4521	−30	−34	3
57.415	May 31	−41	4	16	Wy	+.140	+.932	1.42	.56	8779	3615	+24	+31	3
.535	July 14	−12	4	16	Fr	−.566	+.802	1.42	.58	8735	3494	+21	+25	2
58.404	May 28	−16	4	16	Po	+.210	+.917	1.38	+.75	8911	2719	−3	−5	2

Bi = Leendert Binnendijk	Ke = Geoffrey Keller	Ro = Karl Hans Roth
Da = John J. Davis, Jr.	Ko = Robert H. Koch	Sl = Morton L. Slater
D = Roy W. Delaplaine	K = Michael S. Kovalenko	T = Armstrong Thomas
Dt = Daniel F. Detwiler	m = Peter A. Morris	k = Peter van de Kamp
fl = Edith Flather	P = John H. Pitman	Wo = Charles E. Worley
Fr = Laurence W. Fredrick	Po = William Poole, Jr.	Wy = Arne A. Wyller
		Y = George B. Yntema

TABLE II. REFERENCE STARS

No.	Name	m_{pv}	Spectrum	Diameter (mm)	x_s (mm)	y_s (mm)	Dep. 1940	ΔD/yr
1	BD +46°2271	10.2	F2	.108	−44.62	+41.22	.220	−.00035
2	+45 2506	9.3	G0	.158	+16.67	−67.00	.294	88
3	+45 2507	10.3	G0	.099	+27.95	+25.78	.486	−53
	Fu 46	(10.0)	M4	.116	+8.60	+1.86		

A combined solution for π and α gives the following results:

	p.e.	Weight
$c_x = +8.59609$ mm		
$c_y = +1.86963$		
$\mu_x = +0.015721 = +0\overset{''}{.}29666$	$\pm\overset{''}{.}00056$	4019
$\mu_y = -0.087319 = -1.64771$	$\pm.00056$	4027
$\pi = +0.008382 = +0.1582$	$\pm.0041$	73.38
$\alpha = +0.002839 = +0.0537$	$\pm.0039$	82.57
p.e. $1 = \pm0.00189 = \pm0.0356$		

Table I gives the residuals from the combined solution for each night.

Discussion. Table III gives the mean epoch for the normal places of residuals in x and in y, together with the total weights and number of nights. Considering the weights of the mean residuals, no significant trend exists. The wide range in separations of the components over the period has produced no obvious distortion in the photocentric orbit.

The present determination of the relative parallax $+\overset{''}{.}1582 \pm \overset{''}{.}0041$ supersedes the earlier Sproul determination which is included in the present investigation. The value remains virtually unaltered. Although the weight of the parallax determination has been increased from 45 to 73, the probable error has not been diminished. The p.e. $1_x = \pm\overset{''}{.}026$ for the earlier study was artificially low; the average Sproul value is $\pm\overset{''}{.}033$ (Lippincott 1957). The p.e. 1_x for the present material yields $\pm0\overset{''}{.}033$, and hence a more realistic value for the p.e. of the parallax.

The usual statistical procedure for determining the parallax of the reference system (Lippincott 1958) has been modified due to the large proper motion of $\overset{''}{.}132$ for reference star no. 2 (Table II). Dr. Vyssotsky's spectra do not permit the classification of luminosity type. In this case, the G0 probably indicates a dwarf star both from the point of view of paucity of G0 stars of greater luminosity and the high tangential velocity indicated even for the case of a dwarf star. Assuming a G0V, a parallax $+\overset{''}{.}01$ is adopted; the average parallax for the other two reference stars is found in the usual Sproul manner, resulting in a weighted mean parallax $+\overset{''}{.}0062$ for the reference system. The value of the absolute parallax of Furuhjelm 46 for the Sproul material 1938–1958 is therefore $+\overset{''}{.}1644 \pm \overset{''}{.}0041$.

Other trigonometric parallax determinations adjusted for the Yale precepts (Jenkins 1952) are

McCormick	$+\overset{''}{.}147 \pm \overset{''}{.}013$ (p.e.)
Yerkes	$+.136 \pm .012$
Mt. Wilson	$+.149 \pm .012$.

TABLE III. NORMAL PLACES OF RESIDUALS

Epoch	Σp	Σn	v_x (unit .0001 mm)	v_y (unit .0001 mm)
1938.82	4	4	+22	− 6
40.99	13	5	+10	− 4
42.90	12	4	− 1	− 2
44.44	13	5	+ 1	− 8
45.42	14	5	−11	+14
46.47	19	7	− 4	0
48.07	20	7	− 9	+ 6
49.60	19	7	+ 4	0
51.80	18	6	− 1	− 8
53.82	14	5	+ 2	+ 4
55.43	15	6	+ 6	− 6
57.28	10	4	+ 2	+ 3

The weighted mean value combined with the current Sproul value (unadjusted for Yale precepts) yields

$$\pi_{abs} = +''.160 \pm ''.003.$$

The value of α from the combined solution leads to

$$\frac{\alpha}{a} = B - \beta = +.076 \pm .006 \text{ (p.e.)}$$

the probable error arising from the error in the α alone. The value given by Eggen (1956) for Δm, .37, leads to $\beta = .417$. Adopting the above P, a and π_{abs}, we find

$$a = 4.44 \pm .08 \text{ a.u.}$$

$$M_A + M_B = .516 \pm .028\odot$$

the p.e. arising from the error in a (a.u.) alone. The individual masses are

$$M_A = .262\odot$$

$$M_B = .254\odot$$

where the probable errors are about $\pm.02\odot$ due to the probable errors in the parallax, the α and the Δm. The absolute visual magnitudes of the two components are 11.01 and 11.39, respectively.

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NOTES AND OBSERVATIONS

NOTE ON ORBITAL ELEMENTS OF DOUBLE STARS

We may define as the elements of the orbit of a double star any set of seven independent parameters which, given the double star's right ascension and declination, fix the orientation of the orbit plane in space, the orientation of the orbit in its plane, the scale of the orbit and the motion of the companion along the orbit as a function of the time.

The conventional elements—period P, epoch of periastron passage T, eccentricity e, semi axis major a, inclination i, (ascending) node Ω and longitude of periastron ω—satisfy these conditions. The well-known fact that, in the absence of observations of the component along the line of sight, it is not possible to distinguish between ascending and descending node is not really of importance in this respect.

Other systems of elements, which satisfy the conditions mentioned, have been suggested. It depends entirely on the purpose for which the elements are needed which system will be found most convenient. If, for example, one wished to study the distribution of the orbit planes, the substitution of the right ascension and declination of the positive pole of the orbit plane for the inclination and node would be advantageous. Or, if one were interested in computing dynamical parallaxes, the substitution of $h_1 = aP^{-2/3}$ for the semi-axis major, and so on.

It would appear that for most if not all purely computational purposes (determination of the orbit, ephemeris, differential corrections, etc.) the most convenient system so far suggested has as the "elements": the mean motion n, T, e and the Thiele-Innes constants A, B, F and G (van den Bos 1926).

D. Belorizky (1959) has given a graphical method for the determination of the ephemeris for visual double stars.

If I understand the author correctly, he there advocates the use of Thiele's constants a, A, b and B (Thiele 1883), which he denotes a_1, p, b_1 and q, in preference to the Thiele-Innes constants A, B, F and G. This would seem to me to be a retrograde step, largely because the Thiele-Innes constants are functions of the four geometrical elements a, i, ω and Ω only, while the Thiele constants are functions of these four *plus* the eccentricity e.

Belorizky states: "These elements are thus not well adapted for the calculation of the elements of the orbit, and particularly for that of the half-major axis. Moreover, one has to turn back to a_1, b_1, p and q for the construction of the apparent orbit. These last quantities are more convenient for the determination of the real orbit (Belorizky 1957), and appear quite naturally in that of the apparent orbit."

For those who prefer geometrical construction to computing, I have given (van den Bos 1932, 1933) a method for constructing the apparent orbit directly from the data A, B, F and G and e, which seems to me simple enough. The easiest way to "construct" the apparent orbit from orbital elements is undoubtedly to plot points from the ephemeris, which one has already had to compute in any case.

As far as I can see, in Belorizky's construction drawing MC parallel to OR would not locate the point C, unless the apparent orbit had already been drawn. Or, alternatively and simpler, one would have to construct the length of MC in a similar way to that of OM, by drawing a semicircle on RO as diameter.

For the conversion of the Thiele-Innes constants we have:

$$A + G = a(1 + \cos i) \cos (\omega + \Omega)$$
$$A - G = a(1 - \cos i) \cos (\omega - \Omega)$$
$$B - F = a(1 + \cos i) \sin (\omega + \Omega)$$
$$-B - F = a(1 - \cos i) \sin (\omega - \Omega)$$

The quadrants of $\omega + \Omega$ and $\omega - \Omega$ are unambiguously determined by the Thiele-Innes constants, as a, $1 + \cos i$ and $1 - \cos i$ cannot be negative. The quotient of the first and third equation determines $\omega + \Omega$, that of the second and fourth $\omega - \Omega$. This still gives two solutions for ω and Ω; we take the one where $\Omega < 180°$.

Having found $\omega + \Omega$ and $\omega - \Omega$, we have

$$a(1 + \cos i) = (A + G) \sec (\omega + \Omega) = (B - F) \operatorname{cosec} (\omega + \Omega)$$
$$a(1 - \cos i) = (A - G) \sec (\omega - \Omega) = (-B - F) \operatorname{cosec} (\omega - \Omega).$$

Adding these gives $2a$, after which dividing either by a gives $\cos i$.

Thiele's solution of the same problem, using Belorizky's notation, is:

$$\gamma \cos G = a_1$$
$$\gamma \sin G = b_1 \sec \varphi \quad (\sin \varphi = e)$$

giving the auxiliary quantities γ and G. Then

$$\cos 2\Gamma \cos 2\omega = \cos 2G$$
$$\cos 2\Gamma \sin 2\omega = -\sin 2G \cos (q - p)$$
$$\sin 2\Gamma = \sin 2G \sin (q - p)$$

giving ω and the auxiliary quantity Γ. Then

$$\cos 2\Gamma \cos (q + p - 2\Omega) = \cos (q - p)$$
$$\cos 2\Gamma \sin (q + p - 2\Omega) = \cos 2G \sin (q - p)$$

giving Ω, when remembering that ω and $p - \Omega$ must be in the same quadrant. Finally, a and i by

$$a = \gamma \cos \Gamma \quad a \cos i = \gamma \sin \Gamma.$$

I entirely fail to see that this process is simpler than the one above, and, therefore, that the Thiele constants are better adapted to the calculation of the elements of the true orbit than the Thiele-Innes constants.

Belorizky's solution (1957) is not directly comparable, as he starts from different data, related to the axes of the apparent orbit. It therefore appears to be distantly related to another method (van den Bos 1927), which is also based on the axes of the apparent orbit.

For the computation of an ephemeris, the comparison is:

Thiele constants

$$x = a_1 \cos p(\cos E - e) + b_1 \cos q \sin E$$
$$y = a_1 \sin p(\cos E - e) + b_1 \sin q \sin E$$

Thiele-Innes constants

$$x = AX + FY$$
$$y = BX + GY.$$

In the first case one has to find E (say, by Åstrand's table), $\sin E$, $\cos E$, $\cos E - e$, $\sin p$,

cos p, sin q and cos q; in the second, X and Y from the X, Y tables. Then, in the first case, six multiplications (a_1 with $\cos E - e$, b_1 with $\sin E$ and the two products with cos p and sin p, cos q and sin q, respectively) against four in the second case. The first process hardly seems simpler than the second. Thiele himself gives as the easiest process:

$$\rho \cos [\theta - \tfrac{1}{2}(p + q)] = (b_1 \sin E + a_1 \cos E - a_1 e) \cos \tfrac{1}{2}(q - p)$$

$$\rho \sin [\theta - \tfrac{1}{2}(p + q)] = (b_1 \sin E - a_1 \cos E + a_1 e) \sin \tfrac{1}{2}(q - p)$$

which does not seem preferable to the Thiele-Innes constants either.

In differential correction of the orbit, the great convenience of the Thiele-Innes constants in computing the equations of condition would, I believe, be entirely lost by returning to the Thiele constants, apart from the fact that all seven unknowns would appear in the equations, whereas in the other case the equation in x does not contain ΔB and ΔG, that in y, ΔA and ΔF. This fact obviously shortens the derivation and solu tion of the normal equations.

The above considerations appear to show tha the substitution of the Thiele constants for th Thiele-Innes constants would be a retrograd step in every respect.

On one point I find myself in complete agree ment with Belorizky. It is that the fact that w may safely assume that both Thiele and Inne were familiar with the Gaussian constants (and presumably, with Lagrange's papers) in no wa detracts from their merit in having shown, fo the first time, how to simplify computations i double star orbit work.

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NOTICE

The National Science Foundation announces that the next closing date for receipt of proposals f support of renovation and/or construction of graduate level (doctoral) research laboratories is Marc 1, 1960. Proposals received prior to that date will be reviewed during late spring and early summe Disposition of approved proposals will be made during late summer 1960. Proposals received afte the March 1, 1960 closing date will be reviewed following the next closing date which is expected t be September 1, 1960.

This program will continue to require at least 50 per cent participation by the institution wi funds derived from non-Federal sources. Proposals may be submitted for modernization or constru tion of research laboratories, including laboratory furnishings but not including apparatus or equi ment, in any field of the natural sciences. For the present, this Program is restricted to those depar ments which have an on-going program leading to the Ph.D degree. Support of facilities to be use primarily for instructional purposes will not be considered. It is suggested that, depending on th discipline involved, preliminary inquiry be made to either the Division of Biological and Medic Sciences or the Division of Mathematical, Physical, and Engineering Sciences, National Scien Foundation, Washington 25, D. C. Information concerning the Program and instructions for prepar tion of proposals may be obtained upon request.